Journal of Functional Analysis 279 (2020) 108735

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Calderón–Zygmund singular operators in extrapolation spaces

Vakhtang Kokilashvili [a], Mieczysław Mastyło [b], Alexander Meskhi [a,c,*]

[a] *Department of Mathematical Analysis, A. Razmadze Mathematical Institute, I. Javakhishvili Tbilisi State University, 6. Tamarashvili Str., Tbilisi 0177, Georgia*
[b] *Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznań, Uniwersytetu Poznańskiego 4, 61-614 Poznań, Poland*
[c] *Faculty of Informatics and Control Systems, Georgian Technical University, 77, Kostava St., Tbilisi 0175, Georgia*

A R T I C L E   I N F O

Article history:
Received 1 April 2020
Accepted 5 August 2020
Available online 11 August 2020
Communicated by Camil Muscalu

MSC:
42B20
42B25
46B70

Keywords:
Extrapolation spaces
Calderón–Zygmund singular operators
Maximal operator
A_p-weights
Orlicz spaces

A B S T R A C T

We study the boundedness of the Hardy–Littlewood maximal operator in abstract extrapolation Banach function lattices and their Köthe dual spaces. The extrapolation spaces are generated by compatible families of Banach function lattices on quasi-metric measure spaces with doubling measure. These results combined with a variant of the integral Coifman–Fefferman inequality imply that every Calderón–Zygmund singular operator is bounded in considered extrapolation spaces. We apply these results to extrapolation spaces determined by compatible families of Calderón–Lozanovskii spaces, in particular to compatible families of Orlicz spaces that are interpolation of weighted L^p-spaces $(1 < p < \infty)$ with A_p weights defined on spaces of homogeneous type.

* Corresponding author.
E-mail addresses: vakhtang.kokilashvili@tsu.ge (V. Kokilashvili), mastylo@amu.edu.pl (M. Mastyło), alexander.meskhi@tsu.ge (A. Meskhi).

https://doi.org/10.1016/j.jfa.2020.108735
0022-1236/

1. Introduction

We study the boundedness of Calderón-Zygmund singular integral operators in general class of extrapolation Banach function lattices over the so-called spaces of homogeneous type (i.e., quasi-metric measure spaces with doubling measure). We note that singular integral operators appear in applications of many areas of analysis, including Harmonic Analysis, complex function theory, and PDE. One of the most important examples of singular operators is the Hilbert transform, which arises in the Fourier analysis, in problems concerning almost everywhere convergence of Fourier series as well as in the potential theory. In the study various operators on different function spaces, maximal functions are used for controlling pointwise estimates of these operators including the Poisson integral operator or other even more complicated singular operators. The Hardy–Littlewood maximal operator in its various forms plays a fundamental role in Harmonic Analysis. We recall that for a locally integrable function f on $\mathbb{R}^n$ the classical Hardy–Littlewood maximal operator is defined by

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int\limits_{Q} |f(y)|\, dy, \qquad x \in \mathbb{R}^n,$$

where the supremum is taken over all the cubes Q with sides parallel to the axes and Q is the volume of $|Q|$.

The most fundamental singular integral operator is the Hilbert transform H. Given a function f in $L^p(\mathbb{R})$ with $1 < p < \infty$, Hf is defined by the principal value integral

$$Hf(x) = \frac{1}{\pi}\text{p.v.} \int\limits_{\mathbb{R}} \frac{f(y)}{x-y}\, dy := \frac{1}{\pi} \lim_{\varepsilon \to 0} \int\limits_{\{|x-y|>\varepsilon\}} \frac{f(y)}{x-y}\, dy, \qquad x \in \mathbb{R}.$$

It is well known that this limit exists in the sense of both L^p norm and pointwise almost everywhere. An n-dimensional analogue of the Hilbert transform are the Riesz transforms R_j, $1 \le j \le n$ given by

$$R_j f(x) := \frac{\Gamma((n+1)/2))}{\pi^{(n+1)/2}} \lim_{\varepsilon \to 0} \int\limits_{\{|x-y|>\varepsilon\}} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy, \qquad x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

for all f in Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

A natural generalization of the Hilbert transform is a Calderón–Zygmund singular integral operator on $\mathbb{R}^n$ with kernels which satisfy decay and smoothness conditions. Recall that a Calderón–Zygmund singular operator T is a linear operator that is bounded in $L^2(\mathbb{R}^n)$ and

$$Tf(x) := \int\limits_{\mathbb{R}^n} K(x,y) f(y)\, dy, \qquad f \in L_c^\infty(\mathbb{R}^n), \;\; x \notin \text{supp} f,$$

where the kernel K satisfies the size and smoothness estimates for some $C > 0$ and $\gamma > 0$,

$$|K(x,y)| \le \frac{C}{|x-y|^n}, \quad x \ne y,$$

and

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \le C\frac{|x-x'|^\gamma}{|x-y|^{n+\gamma}}$$

for all x, x', $y \in \mathbb{R}^n$ with $|x-y| > 2|x-x'|$. We note that every Calderón–Zygmund operator is bounded in $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

Recently there has been interest in the study the boundedness of Calderón–Zygmund singular integral operators and the Hardy–Littlewood maximal operator in various function spaces over spaces of homogeneous type. This has been motivated by the applications to the nonlinear potential theory on metric spaces (see [6]). We note that homogeneous spaces extend Euclidean spaces, and include C^∞ compact Riemannian manifolds and Carleson curves.

Interpolation and extrapolation theories deliver powerful methods in the study of abstract spaces as well as operators on these spaces. Without going into too much details, let us just point out that the general extrapolation theory studies natural limiting spaces associated with various interpolation scales and provides estimates for norms of appropriate operators. Many known constructions or results on boundedness of operators are the by-products of abstract machinery connected with the interpolation or extrapolation methods. For some remarkable applications of theory extrapolation theory, we refer to Jawerth and Milman paper [16] and also to the recent paper by Astashkin and Milman [5]. We mention also the papers [4] and [9] regarding some open problems and latest results about grand Lebesgue spaces and extrapolation.

Nowadays, the theory of grand Lebesgue spaces $L^{p)}$ introduced by Iwaniec and Sbordone [15] is one of the intensively developing directions of the modern analysis. As we will see, these spaces are in fact extrapolation spaces generated by a special family of L_p-spaces. The necessity for the study of these spaces was recognized due to their rather essential role and applications in various fields. In [15] the authors prove that if $f = (f_1, \dots, f_n)\colon \Omega \to \mathbb{R}^n$ belongs to Sobolev space $W^{1,1}$, where Ω is an open subset in $\mathbb{R}^n$ with $n \ge 2$, then the Jacobian determinant of f: $J(f,x) := \det Df(x) \ge 0$ belongs to $L^1_{\text{loc}}(\Omega)$ whenever $g \in L^{n)}$, where $g(x) := |Df(x)|$ is the operator norm of $Df(x)$, $x \in \mathbb{R}^n$.

It turns out that the generalized grand Lebesgue spaces are appropriate in the theory of PDEs for studying the existence and uniqueness of solutions, and, the regularity problems for various nonlinear differential equations. The space $L^{p),\theta}$ (defined on bounded domains in $\mathbb{R}^n$) for arbitrary positive θ was introduced in the paper [12], where the authors study the nonhomogeneous n-harmonic equation div $A(x, \nabla u) = \mu$.

We also note that the boundedness of the Hardy–Littlewood maximal operator in weighted grand spaces $L^{p)}_w([0,1])$, $1 < p < \infty$, which was studied in [10], is equivalent to the fact that the weight belongs to the Muckenhoupt's class $A_p([0,1])$. The same statement is true for the boundedness of the Hilbert transform on $[0,1]$ (see [20]).

The purpose of this work is to establish the boundedness of Calderón-Zygmund singular integral operators in a general class of extrapolation Banach lattices over spaces of homogeneous type. The paper is organized as follows. In Section 2 we provide the definitions and basic facts that will be used in the paper. In Section 3 we investigate the boundedness of the Hardy–Littlewood maximal operator in extrapolation Banach function lattices and their Köthe dual spaces over spaces of homogeneous type. These results, combined with a variant of the integral Coifman–Fefferman inequality, imply that every Calderón–Zygmund singular operator is bounded in considered extrapolation spaces. In the final Section 4, we show the applications to extrapolation spaces generated by compatible families of Calderón–Lozanovskii spaces. We also define general variants of grand spaces generated by a special type of Calderón–Lozanovskii spaces called in literature p-convexification of Banach function lattices. We prove some general results on the boundedness of Calderón–Zygmund singular operators in extrapolation spaces generated by the families of Orlicz spaces with measure having the density weight function in the A_p class for some $1 < p < \infty$. In particular, we prove that in the case of grand Orlicz–Zygmund spaces $L^{p),\theta}_w \log^\alpha L$ over spaces of homogeneous type with integrable weight $w \in A_p$ any Calderón–Zygmund singular operator is bounded in these spaces for all α, $\theta > 0$.

2. Notation and background

Through the paper, we use the standard notation. In particular, given two nonnegative functions f and g defined on a set A, we write $f \prec g$, if there is a constant $c > 0$ such that $f(t) \le cg(t)$ for all $t \in A$, while $f \asymp g$ means that $f \prec g$ and $g \prec f$ hold. If X and Y are topological linear spaces, then $X \hookrightarrow Y$ means that $X \subset Y$ and the inclusion map is continuous. In the case where X and Y are Banach spaces, we write $X \cong Y$ whenever $X = Y$ and the identity map $\mathrm{id}\colon X \to Y$ is an isometry onto.

Let us recall some basic definitions, and begun with the definition of extrapolation spaces which has roots in interpolation theory (see [3], [22, pp. 16-18]). Let $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of Banach spaces. A family $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is said to be *compatible* if there exists a Hausdorff topological vector space $\mathcal{X}$ such that $X_\alpha \hookrightarrow \mathcal{X}$ for all $\alpha \in \mathcal{A}$.

For given two compatible families $\{X_\alpha\} := \{X_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{Y_\alpha\} := \{Y_\alpha\}_{\alpha\in\mathcal{A}}$ with $X_\alpha \subset \mathcal{X}$ and $Y_\alpha \subset \mathcal{Y}$, we write $T\colon \{X_\alpha\} \to \{Y_\alpha\}$ if T is a linear mapping with domain $\mathrm{D}(T) \subset \mathcal{X}$ and such that its restriction $T|_{X_\alpha}\colon X_\alpha \to Y_\alpha$ is bounded for every $\alpha \in \mathcal{A}$, and moreover, $\sup_{\alpha\in\mathcal{A}} \|T\|_{X_\alpha \to Y_\alpha} < \infty$. Banach spaces $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$ are called *extrapolation spaces* with respect to the families $\{X_\alpha\}$ and $\{Y_\alpha\}$ of compatible spaces if the condition $T\colon \{X_\alpha\} \to \{Y_\alpha\}$ implies that T is bounded from X into Y.

By an *extrapolation method* $\mathcal{E}$, we mean a functor defined on a class of compatible families such that $\mathcal{E}(\{X_\alpha\})$ and $\mathcal{E}(\{Y_\alpha\})$ are extrapolation spaces for all compatible families $\{X_\alpha\}$ and $\{Y_\alpha\}$ from that class.

The most important extrapolation methods are the functors of the *sum* and the *intersection* of families of Banach spaces.

Let $\{X_\alpha\} := \{X_\alpha\}_{\alpha\in\mathcal{A}}$ be a compatible family of Banach spaces such that there exists a Banach space $X \subset X_\alpha$ for all $\alpha \in \mathcal{A}$ and $\sup_{\alpha\in\mathcal{A}} \|\mathrm{id}\colon X \to X_\alpha\| < \infty$, we let

$$\Delta(\{X_\alpha\}) := \Big\{x \in \bigcap_{\alpha\in\mathcal{A}} X_\alpha;\ \|x\|_{\Delta(X_\alpha)} := \sup_{\alpha\in\mathcal{A}} \|x\|_{X_\alpha} < \infty\Big\}.$$

Then $(\Delta(\{X_\alpha\}), \|\cdot\|_{\Delta(X_\alpha)})$ is a Banach space with the following properties:

(a) $\sup_{\alpha\in\mathcal{A}} \|\mathrm{id}\colon \Delta(\{X_\alpha\}) \to X_\alpha\| \le 1$.
(b) If F is a Banach space such that $F \hookrightarrow X_\alpha$ and $c = \sup_{\alpha\in\mathcal{A}} \|\mathrm{id}\colon F \to X_\alpha\| < \infty$, then $\|\mathrm{id}\colon F \to \Delta(\{X_\alpha\})\| \le c$.

In the case where $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is such that there exists a Banach space Y with $X_\alpha \hookrightarrow Y$ and $\sup_{\alpha\in\mathcal{A}} \|\mathrm{id}\colon X_\alpha \to Y\| < \infty$, then we let $\Sigma(\{X_\alpha\})$ to be a space of all $x \in \mathcal{X}$ representable in the form

$$x = \sum_{\alpha\in\mathcal{A}} x_\alpha \quad (x_\alpha \in X_\alpha), \quad \text{where} \quad \sum_{\alpha} \|x_\alpha\|_{X_\alpha} < \infty. \tag{$*$}$$

It follows from the last condition that there are only countably many summands in $\Sigma_{\alpha\in\mathcal{A}}\, x_\alpha$ different from zero. Note also that our hypothesis implies that the series $\Sigma_{\alpha\in\mathcal{A}}\, x_\alpha$ converges absolutely in Y. We equip the space $\Sigma(X_\alpha)$ with the norm

$$\|x\|_{\Sigma(\{X_\alpha\})} := \inf\Big\{\sum_{\alpha\in\mathcal{A}} \|x_\alpha\|_{X_\alpha};\ x = \sum_{\alpha\in\mathcal{A}} x_\alpha\Big\},$$

where the infimum is taken over all possible representations of x in the form $(*)$. Then $\Sigma(\{X_\alpha\})$ is the smallest Banach space with the property $X_\alpha \hookrightarrow \Sigma(X_\alpha)$ for every $\alpha \in \mathcal{A}$.

Throughout the paper for simplicity of notation, we often write $\Delta(X_\alpha)$ and $\Sigma(X_\alpha)$ shortly instead of $\Delta(\{X_\alpha\})$ and $\Sigma(\{X_\alpha\})$ whenever $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is a fixed compatible family of Banach spaces.

We consider functors Δ and Σ generated by families of Banach function lattices. In what follows we use standard notation from Banach space theory and operator theory. If (Ω, Σ, μ) is a σ-finite measure space, $L^0(\mu) = L^0(\Omega, \mu)$ denotes the space of (equivalence classes of) μ-measurable real-valued functions. As usual $L^0(\mu)$ is equipped with the topology τ_μ of convergence in measure on μ-finite sets. A linear subspace X of $L^0(\mu)$ is called an (order) ideal whenever $f \in X$, $g \in L^0(\mu)$, and $|g| \le |f|$ μ-a.e., imply that $g \in X$. An order ideal $X \subset L^0(\mu)$ equipped with a monotone norm $\|\cdot\|$ is called a normed function lattice in $L^0(\mu)$. If the normed (function) lattice X is norm complete and there exists $u \in X$ such that $u > 0$ on Ω, then X is called a Banach function lattice in $L^0(\mu)$. A Banach lattice X is said to have the *Fatou property* (resp., weak Fatou property) if its unit ball B_X is closed in $L^0(\mu)$ (resp., in $(B_X, \tau_\mu|_{B_X})$). It is well known that the *Fatou property* (resp., the *weak Fatou property*) is equivalent to the following property: for any $f \in L^0(\mu)$ and a sequence $f_n \in X$ such that $0 \le f_n \le f$, $f_n \uparrow f$ a.e. and $\sup \|f_n\|_E < \infty$, we have that $f \in X$ and $\|f_n\|_X \to \|f\|_X$ (resp., if $f_n, f \in X$, $0 \le f_n \le f$, $f_n \uparrow f$ a.e., then $\|f_n\|_X \to \|f\|_X$).

We note that any family $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ of Banach function lattices in $L^0(\mu)$ forms a compatible family by any Banach lattice $X \hookrightarrow L^0(\mu)$ (see, e.g., [18,22]).

The Köthe dual space (or associate space) X' of a Banach function lattice X on (Ω, Σ, μ) is defined as the space of all $f \in L^0$ such that $\int_\Omega |fg|\, d\mu < \infty$ for every $g \in X$. It is a Banach function space in $L^0(\mu)$ equipped with the norm

$$\|f\|_{X'} = \sup_{g \in B_X} \Big| \int_\Omega fg\, d\mu \Big|.$$

Without further references we use the well known fact that a Banach function lattice $X \subset L^0(\mu)$ has the weak Fatou property if and only if (see, e.g., [23] or [18])

$$\|f\|_X = \sup_{g \in B_{X'}} \Big| \int_\Omega fg\, d\mu \Big|, \qquad f \in X.$$

We consider Banach lattices on a quasi-metric measure space with doubling measure. Let (Ω, d) be a quasi-metric (that is, d satisfies the axioms of a metric except for the triangle inequality, which holds in the weaker form $d(x,y) \le \kappa(d(x,z) + d(z,y))$ for some $\kappa \ge 1$) and a positive measure μ that is defined on the σ-algebra generated by quasi-metric balls and open sets. We say that (Ω, d, μ) is a *space of homogeneous type* (SHT for short) if there exists a constant $D_\mu \ge 1$ such that for any $x \in X$ and any $r > 0$,

$$\mu(B(x, 2r)) \le D_\mu\, \mu(B(x,r)),$$

where $B(x,r) := \{y \in \Omega;\ d(x,y) < r\}$ is the ball centered at x with radius r. The family of all balls in (Ω, d) is denoted by $\mathcal{B}$. To avoid trivial measures, we always assume that $0 < \mu(B) < \infty$ for every ball $B \in \mathcal{B}$. Consequently, μ is a σ-finite measure. We also assume that μ is a complete measure.

We also recall some fundamental definitions and results concerning the classes of weights that are related to our investigation. Let (Ω, d, μ) be an SHT and let $w \in L^1(\Omega, \mu)$ be a weight function. We say that w belongs to the class $A_p(\Omega)$ (A_p shortly) for $1 < p < \infty$ if it satisfies the condition:

$$[w]_{A_p} := \sup_{B \in \mathcal{B}} \Big(\frac{1}{\mu(B)} \int_B w\, d\mu \Big) \Big(\frac{1}{\mu(B)} \int_B w^{-\frac{1}{p-1}}\, d\mu \Big)^{p-1} < \infty.$$

A weight w is said to belong to the class A_1 if there exists a constant $C \geq 1$ such that

$$Mw \leq Cw \quad \mu\text{-a.e.},$$

where M is the Hardy-Littlewood maximal operator defined for every $f \in L^1_{\mathrm{loc}}(\Omega, \mu)$ (i.e., f is integrable over all balls B in Ω) by

$$Mf(x) := \sup_{B \in \mathcal{B}} \frac{\chi_B(x)}{\mu(B)} \int_B |f| \, d\mu, \qquad x \in \Omega.$$

The smallest possible constant C is denoted by $[w]_{A_1}$, i.e.,

$$[w]_{A_1} := \operatorname{ess\,sup}_{x \in \Omega} \frac{Mw(x)}{w(x)}.$$

Following [11], we also consider the class A'_∞ of weights w such that

$$[w]_{A'_\infty} := \sup_B \frac{1}{\mu(B)} \int_B M(w\chi_B) \, d\mu < \infty.$$

3. Calderón–Zygmund operators in extrapolation spaces

In this section we investigate the boundedness of maximal and Calderón–Zygmund singular operators in extrapolation Banach function lattices on spaces of homogenous type.

Before starting with the proofs, we recall that if (Ω, d, μ) is a space of homogenous type, then $K\colon \Omega \times \Omega \setminus \{x = y\} \to \Omega$ is a *Calderón–Zygmund kernel* if there exist $\eta > 0$ and $C > 0$ such that it satisfies the decay condition:

$$|K(x_0, y)| \leq \frac{C}{\mu(x_0, d(x_0, y))}$$

for all $x_0 \neq y \in \Omega$, $x_0 \in \Omega$, $x \in \Omega$, and the smoothness condition:

$$|K(x, y) - K(x_0, y)| \leq \left(\frac{d(x, x_0)}{d(x_0, y)}\right)^\eta \frac{1}{\mu(x_0, d(x_0, y))},$$
$$|K(y, x) - K(y, x_0)| \leq \left(\frac{d(x, x_0)}{d(x_0, y)}\right)^\eta \frac{1}{\mu(x_0, d(x_0, y))}$$

for $d(x_0, x) \leq \eta d(x_0, y)$.

Let T be a singular integral operator associated with Calderón–Zygmund kernel K. In addition, if T is bounded in $L^2(\mu)$, then T is a *Calderón–Zygmund singular operator*.

We need a variant of Coifman–Fefferman type inequality, which estimates Calderón–Zygmund operator from above by a maximal function.

Proposition 3.1. *Let T be a Calderón–Zygmund operator on an SHT (Ω, d, μ) and let $1 < p < \infty$. If $w \in A_p(\Omega)$, then for all $f \in L^1_{\mathrm{loc}}(\Omega, \mu)$,*

$$\int_\Omega |Tf| w \, d\mu \le C[w]_{A_p} \int_\Omega (Mf) w \, d\mu,$$

where C is a constant that depends on Ω, μ and T.

Proof. By using the formula of Lerner proved in the homogeneous setting for dyadic maximal operator $M^{\mathcal{D}}$, in [2] it was shown in (see [1, Prop. 4.1]) that, for all $f \in L^1_{\mathrm{loc}}(\Omega, \mu)$, we have

$$\int_\Omega |Tf| w \, d\mu \le C_1[w]_{A_p} \int_\Omega (M^{\mathcal{D}} f) w \, d\mu,$$

where C_1 is an absolute constant that, generally speaking, depends on T. To get the desired result, it is enough to apply the following pointwise inequality which concerns the maximal operator M and its dyadic counterpart $M^{\mathcal{D}}$ (see [13, Prop. 7.9]),

$$M^{\mathcal{D}} f(x) \le C_2 M f(x), \quad x \in \Omega, \quad f \in L^1_{\mathrm{loc}}(\Omega, \mu),$$

where the positive constant $C_2 = C_2(\Omega, \mu, T)$ depends only on Ω, μ and T. □

The following Lemma is indeed well known in the case of Banach function spaces over $\mathbb{R}^n$ with Lebesgue measure; based on Proposition 3.1, we include a proof for the sake of completeness in the setting of Banach function lattices over spaces of homogeneous type.

Lemma 3.2. *Let (Ω, d, μ) be an SHT and let $X \subset L^1_{\mathrm{loc}}(\Omega, \mu)$ be a Banach function lattice with the weak Fatou property. Suppose that the maximal operator M is bounded in X and X'. Then Calderón–Zygmund singular integral operator T is bounded in X with*

$$\|T\|_X \le C \|M\|_X \|M\|_{X'},$$

where C is a constant that depends on Ω, μ and also on T.

Proof. We use the well known Rubio de Francia iteration algorithm to construct A_1 weight with some properties, which also works in the setting of general Banach function lattices. Define a mapping R on X' by the formula:

$$Rg := \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{M^k g}{(\|M\|_{X'})^k}, \quad g \in X',$$

where, for each $k \geq 1$, M^k denotes the k-th iteration of the maximal operator M and $M^0 g := |g|$.

Following the standard proof due to Rubio de Francia, we deduce (by boundedness of M in X') that R is bounded in X' with $\|Rg\|_{X'} \leq 2\|g\|_{X'}$, $|g| \leq Rg$ and $Rh \in A_1$ with

$$[Rg]_{A_1} \leq 2\|M\|_{X'}.$$

Fix $\varepsilon > 0$. Since X has the weak Fatou property, for every $f \in X$ there exists $g \in B_{X'}$ such that

$$\|Tf\|_X \leq (1+\varepsilon) \int_\Omega |Tf||g| \, d\mu.$$

The Rubio de Francia iteration algorithm combined with Proposition 3.1 yields (by the above properties of Rg in A_1 and the fact that $[Rg]_{A_p} \leq [Rg]_{A_1}$ for all $1 < p < \infty$):

$$\begin{aligned} \|Tf\|_X &\leq (1+\varepsilon) \int_\Omega |Tf| Rg \, d\mu \leq c(1+\varepsilon)[Rg]_{A_1} \int_\Omega (Mf) Rg \, d\mu \\ &\leq 2c(1+\varepsilon)\|M\|_{X'} \int_\Omega (Mf) Rg \, d\mu, \end{aligned}$$

where c is a constant that depends on Ω, μ and on T. Consequently, we obtain that for all $f \in X$,

$$\|Tf\|_X \leq 2c(1+\varepsilon)\|Mf\|_X \|Rg\|_{X'} \leq C(1+\varepsilon)\|M\|_X \, \|M\|_{X'} \|f\|_X,$$

where $C = 4c$. Since $\varepsilon > 0$ was arbitrary, the required statement follows. □

We need the following proposition.

Proposition 3.3. *Let $X \subset L^0(\Omega_1, \Sigma_1, \mu)$ and $Y \subset L^0(\Omega, \Sigma_2, \nu)$ be Banach function lattices. Suppose that a mapping S with a domain $D(S) \supset X''$ is such that $|Sf| \leq S|f|$ a.e. for all $f \in X$ and $0 \leq Sg_n \uparrow Sg$ a.e. for any sequence (g_n) in X with $0 \leq g_n \uparrow g$ a.e. If S is bounded from X to Y with $\|Sf\|_Y \leq C\|f\|_X$ for all $\in X$, then S is also bounded from X'' to Y'' with $\|Sf\|_{Y''} \leq C\|f\|_{X''}$ for all $f \in X''$.*

Proof. We apply the following well known fact (cf. [29, pp. 451, 471]): If E is a Banach function lattice, then $g \in E''$ if and only if there exists a sequence (g_n) of elements of E, such that $0 \leq g_n \uparrow |g|$ a.e. and $\sup_{n \geq 1} \|g_n\|_E < \infty$. Moreover for $g \in E''$ we have

$$\|g\|_{E''} = \inf \Big\{ \lim_{n\to\infty} \|g_n\|_E ; \, 0 \leq g_n \uparrow g \text{ a.e.} \Big\}.$$

Next, we fix $f \in X''$, and take any sequence (f_n) in X, such that $0 \le f_n \uparrow |f|$ a.e. and $\sup_n \|f_n\|_X < \infty$. Then our hypotheses imply that $0 \le Sf_n \uparrow Sf$ a.e. (by $Y \hookrightarrow Y''$ with $\|\mathrm{id}\colon Y \to Y''\| \le 1$) and

$$\sup_{n\ge 1} \|Sf_n\|_{Y''} \le \sup_{n\ge 1} \|Sf_n\|_Y \le C \sup_n \|f_n\|_X = C \lim_{n\to\infty} \|f_n\|_X.$$

Clearly, Y'' has the Fatou property. Thus $S|f| \in Y''$ and, we get (by $|Sf| \le S|f|$ a.e.)

$$\|Sf\|_{Y''} \le \|S|f|\|_{Y''} = \lim_{n\to\infty} \|Sf_n\|_{Y''} \le C \lim_{n\to\infty} \|f_n\|_X.$$

Since (f_n) in X was arbitrary, the required estimate follows from the mentioned fact. $\square$

We have the following corollary.

Corollary 3.4. *Let (Ω, d, μ) be an SHT and let $X \subset L^1_{\mathrm{loc}}(\Omega,\mu)$ be a Banach function lattice. Suppose that the maximal operator M is bounded in X with $\|M\|_X \le C$. Then M is also bounded in X'' with $\|M\|_{X''} \le C$.*

Proof. Observe that for any sequence (f_n) in $L^1_{\mathrm{loc}}(\Omega,\mu)$ with $0 \le f_n \uparrow f$ a.e., it follows from the Lebesgue's monotone convergence theorem that

$$\begin{aligned}\lim_{n\to\infty} M(f_n)(x) &= \sup_{n\ge 1}\left(\sup_{B\in\mathcal{B}} \frac{\chi_B(x)}{\mu(B)} \int_B |f_n|\,d\mu\right)\\ &= \sup_{B\in\mathcal{B}}\left(\sup_{n\ge 1} \frac{\chi_B(x)}{\mu(B)} \int_B |f_n|\,d\mu\right) = Mf(x),\end{aligned}$$

for all $x \in \Omega$. Since the domain $D(M) \supset X''$ and $Mf = M|f|$ for all $f \in D(M)$, Proposition 3.3 can be applied. $\square$

We are now ready to state extrapolation theorem for Calderón–Zygmund singular integral operators.

Theorem 3.5. *Let (Ω, d, μ) be an SHT and let $\{X_\alpha\} := \{X_\alpha\}_{\alpha\in\mathcal{A}}$ be a compatible family of Banach function lattices with the Fatou property, such that $X_\alpha \subset L^1_{\mathrm{loc}}(\Omega,\mu)$ for every $\alpha \in \mathcal{A}$. Suppose that the Hardy–Littelwood maximal operator $M\colon \{X_\alpha\} \to \{X_\alpha\}$ and $M\colon \{X'_\alpha\} \to \{X'_\alpha\}$. Then every Calderón–Zygmund singular integral operator is bounded in the extrapolation space $\Delta(\{X_\alpha\})$.*

Proof. It is easy to check that for any $f \in \Sigma(\{X'_\alpha\})$, we have

$$\|f\|_{\Sigma(\{X'_\alpha\})} = \inf\left\{\sum_{\alpha\in\mathcal{A}} \|f_\alpha\|_{X'_\alpha};\ |f| \le \sum_{\alpha\in\mathcal{A}} |f_\alpha| \ \textit{a.e.}\right\}.$$

This implies that M is bounded in $\Sigma(X'_\alpha)$. Then by Corollary 3.4, it follows that M is bounded in $\Sigma(\{X'_\alpha\})''$. Combining the Köthe duality formula $\Delta(\{X'_\alpha\}) \cong \Sigma(\{X_\alpha\})'$ (see [26, Lemma 1]) with the Fatou property of X_α for all $\alpha \in \mathcal{A}$ yields

$$\Delta(\{X_\alpha\})' \cong \Delta(\{(X'_\alpha)'\})' \cong \Sigma(\{X'_\alpha\})''.$$

Thus, we conclude that M is bounded in $\Delta(\{X_\alpha\})'$.

Clearly, M is bounded in $\Delta(\{X_\alpha\})$ and $\Delta(\{X_\alpha\})$ has the Fatou property. To complete the proof, we apply Lemma 3.2 to $X := \Delta(\{X_\alpha\})$. □

4. Applications to Calderón–Lozanovskii extrapolation spaces

We define variants of grand spaces generated by a special type of Calderón–Lozanovskii spaces called the p-convexification of Banach function lattices in literature. We start with some basic definitions related to these spaces. Let $\vec{X} = (X_0, X_1)$ be a couple of Banach function lattices on a measure space (Ω, Σ, μ) and $\mathcal{U}$ denote the set of all non-negative, concave and positively homogeneous continuous functions $\varphi\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ such that $\varphi(0,0) = 0$. Then the Calderón–Lozanovskii construction or the Calderón–Lozanovskii spaces $\varphi(\vec{X}) = \varphi(X_0, X_1)$ consists of all $f \in L^0(\mu)$ such that $|f| \le \lambda\varphi(|f_0|, |f_1|)$ for some $\lambda > 0$ and $f_j \in X_j$ with $\|f_j\|_{X_j} \le 1$, $j = 0, 1$. The spaces $\varphi(\vec{X})$ are Banach ideal spaces on Ω equipped with the norm

$$\|f\|_{\varphi(\vec{X})} := \inf\left\{\lambda > 0;\ |f| \le \lambda\varphi(|f_0|, |f_1|),\ \|f_0\|_{X_0} \le 1,\ \|f_1\|_{X_1} \le 1\right\}$$

(see [24]). In the case of power functions $\varphi(s,t) = s^{1-\theta}t^\theta$ for $s, t > 0$ with $0 < \theta < 1$, the corresponding spaces $\varphi(\vec{X})$ are the well known Calderón spaces $X_0^{1-\theta}X_1^\theta$ (see [8]). In the particular case, we have $X^{1-\theta}(L^\infty)^\theta \cong X^{(p)}$ with $p = 1/(1-\theta)$, where $X^{(p)}$ is known as the p-convexification of X (see [23, p. 53]) equipped with the norm

$$\|f\|_{X^{(p)}} = \big\||f|^p\big\|_X^{1/p}, \quad f \in X^{(p)}.$$

In what follows $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{Y_\alpha\}_{\alpha\in\mathcal{A}}$ are families of Banach lattices and S is a mapping (not necessarily linear) such that $S\colon X_\alpha \to Y_\alpha$ with $\sup_{\alpha\in\mathcal{A}} \|S\|_{X_\alpha\to Y_\alpha} < \infty$, where as usual $\|S\|_{X_\alpha\to Y_\alpha} := \sup_{x\in B_{X_\alpha}} \|Sx\|_{Y_\alpha}$.

It is well known and easy to verify that Calderón–Lozanovskii construction interpolates positive operators (see [28]). The similar proof verifies that if a positive sublinear $T\colon X_0 + X_1 \to Y_0 + Y_1$ such that $T|_{X_j}\colon X_j \to Y_j$ is bounded for both $j = 0$ and $j = 1$, then T is bounded from $\varphi(X_0, X_1)$ to $\varphi(Y_0, Y_1)$ with

$$\|T\|_{\varphi(X_0,X_1)\to\varphi(Y_0,Y_1)} \le \max_{j=0,1} \|T|_{X_j}\|_{X_j\to Y_j}.$$

We are ready to state and prove the result on the boundedness of the Calderón–Zygmund singular operator in extrapolation spaces generated by an extrapolation functor Δ defined for compatible families of Calderón–Lozanovskii spaces over SHT.

Theorem 4.1. *Let (Ω, d, μ) be an SHT and let $\{X_\alpha^j\} := \{X_\alpha^j\}_{\alpha\in\mathcal{A}}$ be a compatible family of Banach function lattices with the Fatou property, such that $X_\alpha^j \subset L^1_{\mathrm{loc}}(\Omega,\mu)$ for all $\alpha \in \mathcal{A}$ and each $j \in \{0,1\}$. Suppose that the corresponding Hardy–Littlewood maximal operator is bounded, $M\colon \{X_\alpha^j\} \to \{X_\alpha^j\}$ and $M\colon \{(X_\alpha^j)'\} \to \{(X_\alpha^j)'\}$ for $j \in \{0,1\}$. Then any Calderón–Zygmund singular integral operator T is bounded in $\Delta\big(\{\varphi_\alpha(X_\alpha^0, X_\alpha^1)\}\big)$ for any family $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ of functions in $\mathcal{U}$ with $\sup_{\alpha\in\mathcal{A}} \varphi_\alpha(1,1) < \infty$.*

Proof. Since the inclusion maps $\mathrm{id}_\alpha\colon X_\alpha^0 \cap X_\alpha^1 \to \varphi_\alpha(X_\alpha^0, X_\alpha^1)$ are bounded with

$$\sup_{\alpha\in\mathcal{A}} \|\mathrm{id}_\alpha\| \le \sup_{\alpha\in\mathcal{A}} \varphi_\alpha(1,1),$$

it follows that $\big\{\varphi_\alpha(X_\alpha^0, X_\alpha^1)\big\}_{\alpha\in\mathcal{A}}$ forms a compatible family of Banach lattices. Next observe that our hypotheses combined with the interpolation property of Calderón–Lozanovskii spaces yields that the maximal operator M is bounded in $\varphi_\alpha(X_\alpha^0, X_\alpha^1)$ for all $\alpha \in \mathcal{A}$ with

$$\sup_{\alpha\in\mathcal{A}} \|M\|_{\varphi_\alpha(X_\alpha^0, X_\alpha^1)} \le \max\Big\{\sup_{\alpha\in\mathcal{A}} \|M\|_{X_\alpha^0},\, \sup_{\alpha\in\mathcal{A}} \|M\|_{X_\alpha^1}\Big\} < \infty.$$

Hence M is bounded in the extrapolation space $\Delta\big(\{\varphi_\alpha(X_\alpha^0, X_\alpha^1)\}\big)$.

Now we use Köthe duality result due to Lozanovskii ([24]), which states that for any couple (X,Y) of Banach function lattices on measure space $(\mathcal{S},\Sigma,\mu)$ and for any $\varphi \in \mathcal{U}$, we have $\varphi(X,Y)' = \widehat{\varphi}(X',Y')$, with universal constants of equivalence norms that do not depend on φ and (X,Y). More precisely we have

$$\|\cdot\|_{\widehat{\varphi}(X',Y')} \le \|\cdot\|_{\varphi(X,Y)'} \le 2\|\cdot\|_{\widehat{\varphi}(X',Y')}.$$

Here $\widehat{\varphi} \in \mathcal{U}$ is an involution of φ defined by

$$\widehat{\varphi}(s,t) = \inf\Big\{\frac{as+bt}{\varphi(a,b)};\ a,b>0\Big\}, \qquad s,t \ge 0.$$

This duality result combined with our hypothesis $M\colon \{(X_\alpha^j)'\} \to \{(X_\alpha^j)'\}$ for both $j=0$ and $j=1$ implies that

$$M\colon \big\{\varphi_\alpha(X_\alpha^0,\, X_\alpha^1)'\big\} \to \big\{\varphi_\alpha(X_\alpha^0,\, X_\alpha^1)'\big\}.$$

To conclude we observe that the proof of Lemma 4.1 in [17] yields that, for any couple (X,Y) of Banach function lattices and $\varphi \in \mathcal{U}$, $\varphi(X,Y)$ has the weak Fatou (resp., Fatou) property whenever both X and Y have the weak Fatou (resp., Fatou) property. To finish it is enough to apply Theorem 3.5 for the family $\{X_\alpha\}_{\alpha\in\mathcal{A}} := \big\{\varphi_\alpha(X^0_\alpha, X^1_\alpha)\big\}_{\alpha\in\mathcal{A}}$. □

We also recall the definition of Orlicz spaces which are strictly connected with Calderón-Lozanovskii spaces. Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a non-decreasing, convex and left-continuous function, not identical 0 on $(0,\infty)$, with $\Phi(0) = 0$. Let $\varphi \in \mathcal{U}$ be defined by $\varphi(s,t) = t\Phi^{-1}(s/t)$ if $t > 0$ and 0 if $t = 0$, where Φ^{-1} is the right continuous inverse of Φ. Then, for every Banach function lattice E in $L^0(\Omega,\Sigma,\mu)$, the Calderón-Lozanovskii space $\varphi(E, L^\infty)$ coincides isometrically with the space

$$E_\Phi := \big\{f \in L^0(\mu);\, \Phi(|f|/\lambda) \in E \text{ for some } \ \lambda > 0\big\}$$

equipped with the norm

$$\|f\|_{E_\Phi} := \inf\big\{\lambda > 0;\, \|\Phi(|f|/\lambda)\|_E \le 1\big\}.$$

In particular, if $E := L^1(\mu)$, we recover the Orlicz space $L_\Phi(\mu)$ (L_Φ shortly).

In what follows for a given Banach space $(X, \|\cdot\|_X)$ and $t > 0$, we let tX to be the Banach space X equipped with the norm $t\,\|\cdot\|_X$. Now we are ready to give the definition of general variants of grand spaces.

Let $p \in (1,\infty)$ and let E be a Banach function lattice in $L^0(\Omega,\Sigma,\mu)$. Assume that $\omega\colon (0, p-1) \to (0,\infty)$ is such a function that $\{\omega(\varepsilon)E^{(p-\varepsilon)}\}_{\varepsilon\in(0,p-1)}$ forms a compatible family of Banach spaces. Then the extrapolation space $\Delta(\{\omega(\varepsilon)E^{(p-\varepsilon)}\}_{\varepsilon\in(0,p-1)})$ is called a *grand space* (on (Ω,Σ,μ)) and is denoted by $\omega E^{p)}$. From our discussion in the previous section, we know that $\omega E^{p)}$ is a Banach function lattice in $L^0(\mu)$ equipped with the norm

$$\|f\|_{\omega E^{p)}} = \sup_{\varepsilon\in(0,p-1)} \omega(\varepsilon)\|f\|_{E^{(p-\varepsilon)}}, \qquad f \in \omega E^{p)}.$$

If w is a weight on Ω, then the grand space defined with respect to the measure $wd\mu$ is denoted by $\omega E^{p)}_w$ (or by $\omega E^{p)}(wd\mu)$). Whenever we mention the grand space $\omega E^{p)}$, we always assume that $\{\omega(\varepsilon)E^{(p-\varepsilon)}\}_{\varepsilon\in(0,p-1)}$ forms a compatible family.

We note that if $\psi\colon (0, p-1) \to (0,\infty)$ is a continuous function that is non-decreasing on $(0,\varepsilon_0]$ for some $\varepsilon_0 < p-1$, and satisfies the condition $\psi(s) \to 0$ as $s \to 0+$ and $\omega(\varepsilon) = \psi(\varepsilon)^{1/(p-\varepsilon)}$ for all $\varepsilon \in (0, p-1)$, then $\{\omega(\varepsilon)E^{(p-\varepsilon)}\}_{\varepsilon\in(0,p-1)}$ form a compatible family of Banach spaces. In this case the grand space is denoted by $E^{p),\psi}$. For $\psi(s) = s^\theta$, where θ is a positive number, we denote $E^{p),\psi}$ by $E^{p),\theta}$. In the case $E = L^p(\mu)$ on $\Omega = (0,1)$ or Ω is a domain in $\mathbb{R}^n$ where μ is the Lebesgue measure, we recover the classical grand Lebesgue spaces intensively studied in the recent time (see [21] and references therein).

We need the following lemma.

Lemma 4.2. *Let $1 < p < \infty$ and let E be a Banach lattice in $L^0(\Omega,\Sigma,\nu)$ such that $\chi_\Omega \in E$ and let $\omega E^{p)}$ be a grand space such that $\sup_{\varepsilon\in(\varepsilon_0,p-1)} \omega(\varepsilon) < \infty$. Suppose that S*

is a sublinear operator such that $\sup_{\varepsilon\in(0,\varepsilon_0)} \omega(\varepsilon)\|S\|_{E^{(p-\varepsilon)}} < \infty$*. Then* S *is bounded in the grand space* $\omega E^{p)}$*.*

Proof. We use the well known inequality (see, e.g., [23, Prop. 1.d.2, p. 43], [22, p. 40]) which is true for any Banach lattice E over any measure space and which states that for all x, $y \in E$ and $\theta \in (0,1)$,

$$\big\||x|^{1-\theta}|y|^{\theta}\big\|_E \le \|x\|_E^{1-\theta}\|y\|_E^{\theta}.$$

If $u, v, r \in (1,\infty)$ satisfy $1/r = 1/u + 1/v$, we conclude (by taking $x = |f|^u$ and $y = |g|^v$) that $fg \in E^{(r)}$ for all $f \in E^{(u)}$ and $g \in E^{(v)}$ with

$$\|fg\|_{E^{(r)}} \le \|f\|_{E^{(u)}}\|g\|_{E^{(v)}}.$$

This implies that for $v \in (1,\infty)$ defined by $1/v := 1/r - 1/u$ where $r := p-\varepsilon$ and $u := p-\varepsilon_0$, the following estimate holds:

$$\|f\|_{E^{(p-\varepsilon)}} \le \|f\|_{E^{(p-\varepsilon_0)}}\|\chi_\Omega\|_{E^{(1/v)}}, \qquad f \in E^{(p-\varepsilon_0)}.$$

This combined with $1/v - 1/u \le \alpha := (p-1-\varepsilon_0)/(p-\varepsilon_0)$ yields, in particular, that for all $f \in \omega E^{p)}$,

$$\begin{aligned}\sup_{\varepsilon\in(\varepsilon_0,p-1)} \|f\|_{E^{(p-\varepsilon)}} &\le C(p,\varepsilon_0)\,\omega(\varepsilon_0)\|f\|_{E^{(p-\varepsilon_0)}}\\ &\le C(p,\varepsilon_0) \sup_{\varepsilon\in(0,\varepsilon_0]} \omega(\varepsilon)\|f\|_{E^{(p-\varepsilon)}},\end{aligned}$$

where $C(p,\varepsilon_0) = \big(\sup_{\varepsilon\in[\varepsilon_0,p-1)} \frac{\omega(\varepsilon)}{\omega(\varepsilon_0)}\big) \max\big\{1, \|\chi_\Omega\|_E^{\alpha}\big\}$. As a consequence, it follows that

$$\omega E^{p)} := \Delta\big(\big\{\omega(\varepsilon)E^{(p-\varepsilon)}\big\}_{\varepsilon\in(0,p-1)}\big) = \Delta\big(\big\{\omega(\varepsilon)E^{(p-\varepsilon)}\big\}_{\varepsilon\in(0,\varepsilon_0)}\big)$$

with

$$\|\cdot\|_{\Delta(\{\omega(\varepsilon)E^{(p-\varepsilon)}\}_{\varepsilon\in(0,\varepsilon_0)})} \le \|\cdot\|_{\omega E^{p)}} \le C(p,\varepsilon_0)\|\cdot\|_{\Delta(\{\omega(\varepsilon)E^{(p-\varepsilon)}\}_{\varepsilon\in(0,\varepsilon_0)})}.$$

Clearly, this yields the required statement. □

We demonstrate the applications of obtained results for extrapolation spaces generated by Orlicz spaces. Before we state the first lemma, we recall that a function $\rho\colon [0,\infty) \to [0,\infty)$ is called *quasi-concave* if it is continuous and positive on $(0,\infty)$, and satisfies $\rho(s) \le \max\big\{1, \frac{s}{t}\big\}\rho(t)$ for all $s, t > 0$. We use an obvious fact that every quasi-concave function ρ has a concave majorant $\widetilde{\rho}$ given by $\widetilde{\rho}(t) := \inf_{s>0}\big(1+\frac{t}{s}\big)\rho(s)$, which satisfies $\rho(t) \le \widetilde{\rho}(t) \le 2\rho(t)$ for all $t > 0$.

We need somewhat technical lemmas.

Lemma 4.3. *Let $p \in (1, \infty)$ and let ϕ be an Orlicz function. The following statements are true for an Orlicz function Φ given by $\Phi(s) := \phi(s^p)$ for all $s \geq 0$:*

(i) *The following formula holds:*

$$X_\Phi \cong (X_\phi)^{(p)}.$$

(ii) *There exists $\varphi \in \mathcal{U}$ such that, for any Banach lattice X the following formula holds:*

$$X_\Phi = \varphi(X^{(p)}, L^\infty),$$

where the constants of equivalence of norms are universal and do not depend on p, ϕ and X.

Proof. (i). The proof of this isometrical formula is obvious so we omit it.

(ii). Fix $p \in (1, \infty)$ and let $\Phi(s) := \phi(s^p)$ for all $s \geq 0$. Then $\Phi^{-1}(s) = \phi^{-1}(s)^{1/p}$ for all $s \geq 0$. Since ϕ^{-1} is a concave function, $\rho(s) := \phi^{-1}(s^p)^{1/p}$ is a quasi-concave function. This implies that $\varphi \in \mathcal{U}$, where $\varphi(0,0) := 0$ and

$$\varphi(s,t) := t\widetilde{\rho}(s/t), \quad s,t > 0.$$

We consider two functions $\varphi_0, \varphi_1 \in \mathcal{U}$ given by $\varphi_0(s,t) := s^{1/p}t^{1-1/p}$ for all $s,t \geq 0$ and $\varphi_1(0,0) := 0$ and $\varphi_1(s,t) := t\Phi^{-1}(s/t)$ for all $s,t > 0$. Then we have

$$\varphi(\varphi_0(s,1),1) = \varphi(s^{1/p},1) \asymp \rho(s^{1/p}) = \varphi_1(s,1), \quad s \geq 0$$

with universal constants of equivalence that do not depend on p.

Now we use the special case of the so-called reiteration formulas (see [25]) which states: If $\varphi_0, \varphi_1, \varphi \in \mathcal{U}$, then for any couple (X_0, X_1) of Banach function lattices

$$\varphi\big(\varphi_0(X_0,X_1),X_1\big) = \varphi_1(X_0,X_1)$$

with equivalence of norms depending only on constants of equivalence $\varphi_1(\cdot,1) \asymp \varphi(\varphi_0(\cdot,1),1)$. We use above formula with $X_0 = X$, $X_1 = L^\infty$ and constructed φ to obtain (by $\varphi_0(X, L^\infty) \cong X^{(p)}$)

$$X_\Phi \cong \varphi_1(X, L^\infty) = \varphi(X^{(p)}, L^\infty)$$

with universal constants of equivalence that do not depend on p and X. This completes the proof. □

Corollary 4.4. *Let ϕ be an Orlicz function and let, for a given $p \in (1, \infty)$, Φ be an Orlicz function given by $\Phi(s) = \phi(s^p)$ for all $s \geq 0$. Then, there exists $\varphi \in \mathcal{U}$ such that for any measure space (Ω, Σ, μ) the following formula holds:*

$$L_\Phi(\mu) \cong L_\phi^{(p)} = \varphi\big(L^p(\mu), L^\infty(\mu)\big)$$

with constants of equivalence of norms that do not depend on p.

We also need the following result.

Lemma 4.5. *Let $1 \le p < q < \infty$ and let Φ be an Orlicz function such that $s \mapsto \Phi(s)/s^p$ is non-decreasing function and $s \mapsto \Phi(s)/s^q$ is non-increasing function. Then there exists $\varphi \in \mathcal{U}$ such that, for any measure space (Ω, Σ, μ),*

$$L_\Phi(\mu) = \varphi\big(L^p(\mu), L^q(\mu)\big),$$

with universal constants of equivalence of norms that do not depend on p, q and Φ.

Proof. By conditions imposed on Φ, it follows that $s \mapsto \Phi^{-1}(s)/s^{1/p}$ is a non-increasing and $s \mapsto \Phi^{-1}(s)/s^{1/q}$ is a non-decreasing function. These assumptions easily imply that there exists a quasi-concave function ρ such that

$$\Phi^{-1}(s) = s^{1/p}\rho\big(s^{1/q-1/p}\big), \quad s > 0.$$

(to see this it is enough to take $\rho(s) := s^{r/p}\Phi^{-1}(s^{-r})$ for $s > 0$, where $r := 1/p - 1/q$). Therefore, by setting $\varphi(0,0) := 0$ and $\varphi(s,t) = s\widetilde{\rho}(t/s)$ for all $s, t > 0$, we get that (by $\rho \le \widetilde{\rho} \le 2\rho$)

$$\Phi^{-1}(s) \le \varphi(s^{1/p}, s^{1/q}) \le 2\Phi^{-1}(s) \quad \text{for all } s \ge 0.$$

Hence, for $\vec{X} := (L^p(\mu), L^q(\mu))$, the standard calculations yield (see [27, pp. 460-461]) that $L_\Phi(\mu) = \varphi(\vec{X})$ with $\|\cdot\|_{\varphi(\vec{X})} \le \|\cdot\|_{L_\Phi} \le 2\|\cdot\|_{\varphi(\vec{X})}$. This completes the proof. □

In what follows we use two obvious properties of weights from the class A_p with $1 \le p < \infty$:

(i) If $1 < p < \infty$, then $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$ and $[w^{1-p'}]_{A_{p'}} = [w]_{A_p}^{p'-1}$;
(ii) If $1 \le p < q < \infty$, then $A_p \subset A_q$ and $[w]_{A_q} \le [w]_{A_p}$.

The first property easily follows from the definitions and that second one by the Hölder inequality.

We apply the so-called "openness property" for A_p weights in $L^0(\Omega, d, \mu)$, where (Ω, d, μ) is an SHT. In the remarkable paper [14], the authors explicitly compute an admissible value of $\varepsilon > 0$ such that any A_p weight w belongs to $A_{p-\varepsilon}$. This result states (see, [14, Thm. 1.2]): *If $w \in A_p$ for a given $1 < p < \infty$, then $w \in A_{p-\varepsilon}$ with*

$$\varepsilon = \frac{p-1}{1 + \tau_{k\mu}[\sigma]_{A'_\infty}},$$

where $\tau_{\kappa\mu} = 6(32\kappa^2(\kappa^2+\kappa)^2)^{D_\mu}$ *and* $\sigma = w^{1-p'}$. *Furthermore*

$$[w]_{A_{p-\varepsilon}} \le 2^{p-1}(4\kappa)^{pD_\mu}[w]_{A_p}.$$

We also use the following estimate (see [14, Thm. 1.3]) of the norm of M in $L^p(w\,d\mu)$ where $w \in A_p$,

$$\|M\|_{L^p(w\,d\mu)} \le C_\Omega\left(\frac{1}{p-1}[w]_{A_p}[\sigma]_{A'_\infty}\right)^{1/p},$$

where the constant C_Ω depends only on the doubling constant of the measure μ and the geometric constant κ of the quasi-metric.

Notice that it follows by above estimate and property (ii) that, for every $w \in A_p$ with $1 < p < \infty$, we have $[\sigma]_{A'_\infty} \le c_\Omega[\sigma]_{A_{p'}} = c_\Omega[w]_{A_p}^{p'-1}$, where $c_\Omega \ge 1$ is also a constant that only depends on the doubling constant of the measure μ and the geometric constant κ of the quasi-metric. In consequence, Buckley's type estimate follows (see [7] in the case of Euclidean space $\mathbb{R}^n$):

$$\|M\|_{L^p_w(\mu)\to L^p_w(\mu)} \le C_\Omega(c_\Omega)^{1/p}\left(\frac{1}{p-1}\right)^{1/p}[w]_{A_p}^{\frac{1}{p-1}} \le \tilde{c}_\Omega p'\,[w]_{A_p}^{\frac{1}{p-1}}.$$

The following theorem shows the boundedness of the maximal operator in grand spaces generated by Orlicz spaces over spaces of homogenous type.

Theorem 4.6. *Let* (Ω, d, μ) *be an SHT and let* w *be a weight in* $A_p(\Omega)\cap L^1(wd\mu)$ *for some* $p \in (1,\infty)$. *Then for every Orlicz function* ϕ, *the maximal Hardy–Littlewood operator* M *is bounded in the grand space* $\omega L_\phi(w\,d\mu)^{p)}$ *whenever* ω *is bounded on any subinterval* $[\delta, p-1)$ *of* $(0, p-1)$.

Proof. For simplicity of notation, we define the measure ν by $d\nu := wd\mu$. Based on the above discussion about properties of weights in the class A_p weights, it follows that for any $w \in A_p$ there exists $\varepsilon_0 \in (0, p-1)$ such that

$$[w]_{A_{p-\varepsilon_0}} \le 2^{p-1}(4\kappa)^{pD_\mu}[w]_{A_p}.$$

Moreover, the maximal operator M is bounded in $L^p(\nu)$ and

$$\|M\|_{L^p(\nu)} \le \tilde{c}_\Omega p'\,[w]_{A_p}^{\frac{1}{p-1}}.$$

We consider two cases: $\varepsilon \in (0, \varepsilon_0]$ and $\varepsilon \in (\varepsilon_0, p-1)$. In the first case we have $w \in A_{p-\varepsilon}$ with $1 \le [w]_{A_{p-\varepsilon}} \le [w]_{A_{p-\varepsilon_0}}$. Thus, the maximal operator M is bounded in $L_{p-\varepsilon}(w\,d\mu)$ and, for all $\varepsilon \in (0, \varepsilon_0]$, we have

$$\|M\|_{L^{p-\varepsilon}(\nu)} \leq \tilde{c}_\Omega (p-\varepsilon)' \, [w]_{A_{p-\varepsilon}}^{\frac{1}{p-\varepsilon-1}} \leq \tilde{c}_\Omega (p-\varepsilon_0)' \, [w]_{A_{p-\varepsilon_0}}^{\frac{1}{p-\varepsilon_0-1}}$$
$$= C(\Omega, p, [w]_{A_{p-\varepsilon_0}}),$$

where $C(\Omega, p, [w]_{A_{p-\varepsilon_0}})$ is a constant that does not depend on $\varepsilon \in (0, \varepsilon_0)$. Consequently,

$$M \colon \big\{\omega(\varepsilon) L_\phi(\nu)\big\}_{\varepsilon \in (0,\varepsilon_0)} \to \big\{\omega(\varepsilon) L_\phi(\nu)\big\}_{\varepsilon \in (0,\varepsilon_0)}.$$

By the conditions imposed on w, we have $\chi_\Omega \in L_\phi(\nu)$. Thus, we can apply Lemma 4.2 to obtain that M is bounded in $\omega L_\phi(\nu)^{p)}$. □

We prove the boundedness of Calderón–Zygmund singular operator in extrapolation spaces generated by Orlicz spaces over SHT.

Theorem 4.7. *Let $1 < q < \infty$ and let (Ω, d, μ) be an SHT, and let w be a weight in $A_p(\Omega) \cap L^1(\mu)$ for some $1 < p < q$. Suppose that a family of Orlicz functions $\{\Phi_\varepsilon\}_{\varepsilon \in (0,\, p-1)}$ with $\sup_{\varepsilon \in (0,\, p-1)} \Phi_\varepsilon^{-1}(1) < \infty$ such that $t \mapsto \Phi_\varepsilon / t^{p-\varepsilon}$ is a non-decreasing and $t \mapsto \Phi_\varepsilon(t)/t^q$ is a non-increasing function for all $\varepsilon \in (0,\, p-1)$. Then every Calderón–Zygmund singular operator T is bounded in the extrapolation space $\Delta(\{\omega L_{\Phi_\varepsilon}(w\, d\mu)\}_{\varepsilon \in (0,\, p-1)})$ generated by any function $\omega \colon (0,\, p-1) \to (0, \infty)$ that is bounded on any subinterval $[\delta,\, p-1)$ of $(0, p-1)$.*

Proof. For simplicity of notation, we let $d\nu := w d\mu$. We now apply Lemma 4.5 and its proof to find a family $\{\varphi_\varepsilon\}_{\varepsilon \in (0,p-1)} \subset \mathcal{U}$ such that $\Phi_\varepsilon^{-1}(s) = \varphi_\varepsilon(s^{1/(p-\varepsilon)}, s^{1/q})$ for all $s > 0$ and

$$L_{\Phi_\varepsilon}(\nu) = \varphi\big(L^{p-\varepsilon}(\nu), L^q(\nu)\big),$$

with universal constants of equivalence of norms that do not depend on p, q and ε. This implies that $\sup_{\varepsilon \in (0,\, p-1)} \varphi_\varepsilon(1,1) < \infty$ and moreover $\big\{\omega(\varepsilon)\varphi(L^{p-\varepsilon}(\nu), L^q(\nu))\big\}_{\varepsilon \in (0,p-1)}$ forms an admissible family with

$$\Delta\big(\big\{\omega(\varepsilon) L_{\Phi_\varepsilon}(\nu)\big\}_{\varepsilon \in (0,\, p-1)}\big) = \Delta\big(\big\{\omega(\varepsilon)\varphi(L^{p-\varepsilon}(\nu), L^q(\nu))\big\}_{\varepsilon \in (0,\, p-1)}\big).$$

Now observe that it follows from the proof of Theorem 4.6 that there exists $\varepsilon_0 \in (0, p-1)$ such that

$$\sup_{\varepsilon \in (0,\varepsilon_0)} \|M\|_{L^{p-\varepsilon}(\nu)} < \infty.$$

Since $p - \varepsilon_0 < q$, we have $w \in A_q$ with $[w]_{A_q} \leq [w]_{A_{p-\varepsilon_0}}$,

$$\|M\|_{L^q(\nu)} \leq C(\Omega, p, [w]_{A_p}).$$

Thus, by interpolation property for Calderón–Lozanovskii space, it follows that

$$M\colon \Delta\big(\{\omega(\varepsilon)\varphi_\varepsilon(L^{p-\varepsilon}(\nu), L^q(\nu))\}_{\varepsilon\in(0,\varepsilon_0)}\big) \to \Delta\big(\{\omega(\varepsilon)\varphi_\varepsilon(L^{p-\varepsilon}(\nu), L^q(\nu))\}_{\varepsilon\in(0,\varepsilon_0)}\big).$$

Now observe that Calderón-Lozanovskii space $\varphi(X_0, X_1)$ can be defined as the space of all functions $f \in L^0(\mu)$ such that $|f| = \varphi(|f_0|, |f_1|)$ for some $f_0 \in X_0$, $x_1 \in X_1$ with the norm

$$\|f\|_\varphi = \inf \max\{\|f_0\|_{X_0}, \|f_1\|_{X_1}\},$$

where the infimum is taken over all $f_0 \in X_0$, $f_1 \in X_1$ for which $|f| = \varphi(|f_0|, |f_1|)$. Combining this fact with proof of Lemma 4.2 (for $E = L_1$), we easily get that for all $f \in \Delta(\{L_{\Phi_\varepsilon}(\nu)\})$,

$$\sup_{\varepsilon\in[\varepsilon_0,\, p-1)} \omega(\varepsilon)\|f\|_{\varphi_\varepsilon(L^{p-\varepsilon}(\nu), L^q(\nu))} \le C(p, \varepsilon_0) \sup_{\varepsilon\in(0,\, \varepsilon_0)} \omega(\varepsilon)\|f\|_{\varphi_\varepsilon(L^{p-\varepsilon}(\nu), L^q(\nu))}$$

and whence

$$\Delta\big(\{\omega(\varepsilon)L_{\Phi_\varepsilon}(\nu)\}_{\varepsilon\in(0,\, p-1)}\big) = \Delta\big(\{\omega(\varepsilon)\varphi_\varepsilon(L^{p-\varepsilon}(\nu), L^q(\nu))\}_{\varepsilon\in(0,\, \varepsilon_0)}\big).$$

To complete the proof we recall that by property (i) of weights in the class A_r, for $r \in (1, \infty)$, we have that $w \in A_r$ is equivalent to $w^{1-r'} \in A_{r'}$ with $[w^{1-r'}]_{A_{r'}} = [w]_{A_r}^{r'-1}$. This implies that the maximal operator M is bounded in $L^r(\nu))' \cong (L^r(w\, d\mu))' \cong L^{r'}(w^{1-r'} d\mu)$ with

$$\|M\|_{L^r(\nu)'} \le Cr[w]_{A_r},$$

where $C = C(\mu, \kappa)$. Now if we take $r = p - \varepsilon$ with $\varepsilon \in (0, \varepsilon_0)$, we conclude that

$$\|M\|_{L^{p-\varepsilon}(\nu)'} \le C(p - \varepsilon)[w]_{A_{p-\varepsilon}} \le p[w]_{p-\varepsilon_0} \le \widetilde{C}(\Omega, p, [w]_{A_p}).$$

This shows that $M\colon \{L^{p-\varepsilon}(\nu)'\}_{\varepsilon\in(0,\varepsilon_0)} \to \{L^{p-\varepsilon}(\nu)'\}_{\varepsilon\in(0,\varepsilon_0)}$. Similarly, for $q \in [p, \infty)$, we get (by $w \in A_q$ with $[w]_{A_q} \le [w]_{A_p}$),

$$\|M\|_{L^q(wd\mu)'} \le q[w]_{A_p}.$$

Now, by the Köthe duality formula for Calderón–Lozanovskii spaces, we have for all $\varepsilon \in (0, \varepsilon_0)$,

$$\big(\omega(\varepsilon)\varphi_\varepsilon(L^{p-\varepsilon}(\nu), L^q(\nu)\big)' = \omega(\varepsilon)^{-1}\widehat{\varphi_\varepsilon}(L^{p-\varepsilon}(\nu)', L^q(\nu)')$$

with universal constants of equivalence of norms (that do not depend on ε). To finish, we apply Theorem 4.1 for the family of functions $\{\omega(\varepsilon)\varphi_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)} \subset \mathcal{U}$ to get the required statement. $\square$

By applying Theorem 4.7, we establish the boundedness of Calderón–Zygmund singular operator in extrapolation spaces generated by Orlicz–Zygmund spaces over SHT. We briefly fix some required notation. For $p \in (1, \infty)$ and $\alpha > 0$, we consider an Orlicz function $\Phi(t) = t^p \log^\alpha(1+t)$ for $t \geq 0$. Then Orlicz space L_Φ on any measure space (Ω, Σ, μ) is called an Orlicz–Zygmund space and is denoted by $L^p \log^\alpha L$.

Let $\{\Phi_\varepsilon\}_{\varepsilon \in (0, p-1)}$ be a family of Orlicz functions given by $\Phi_\varepsilon(t) := t^{p-\varepsilon} \log^\alpha(1+t)$ for all $t \geq 0$ and let $\omega(\varepsilon) := \varepsilon^{\theta/(p-\varepsilon)}$ for all $\varepsilon \in (0, p-1)$ and some $\theta > 0$. Then the extrapolation space $\Delta\big(\{\omega(\varepsilon) L_{\Phi_\varepsilon}(w d\mu)\}_{\varepsilon \in (0,p-1)}\big)$ is well defined. In what follows this space is called *grand Orlicz–Zygmund space* and is denoted by $L_w^{p),\theta} \log^\alpha L$.

We note that from a general result on the boundedness of the Hilbert transform H in Orlicz spaces $L_\Phi(w\, dm)$ over $[0,1]$ equipped with the Lebesgue measure m, it follows that H given by

$$Hf(t) = \text{p.v.} \int_0^1 \frac{f(s)}{t-s}\, ds$$

is bounded in the Orlicz–Zygmund space $L^p \log^\alpha L(w\, dm)$ if (and only if) $w \in A_p([0,1])$ (see [19, Thm. 3.4.1, p. 113])).

We have the following version of the theorem for any Calderón–Zygmund singular operator in the case of the grand Orlicz–Zygmund spaces over spaces of homogenous type.

Theorem 4.8. *Let (Ω, d, μ) be an SHT and let w be a weight in $A_p(\Omega) \cap L^1(\mu)$ for some $1 < p < \infty$. Then every Calderón–Zygmund singular operator T is bounded in the grand Orlicz–Zygmund space $L_w^{p),\theta} \log^\alpha L$ over (Ω, d, μ) for all $\alpha,\ \theta > 0$.*

Proof. Let $\{\Phi_\varepsilon\}_{\varepsilon \in (0, p-1)}$ be a family of Orlicz functions which generate our space. Clearly, $t \mapsto \Phi_\varepsilon(t)/t^{(p-\varepsilon)} = \log^\alpha(1+t)$ is an increasing and $t \mapsto \Phi_\varepsilon(t)/t^{(p+\alpha)} = \frac{1}{t^\varepsilon}\big(\frac{\log(1+t)}{t}\big)^\alpha$ is a decreasing function on $(0, \infty)$. Since $\Phi_\varepsilon^{-1}(\log^\alpha 2) = 1$, we have $\sup_{\varepsilon \in (0, p-1)} \Phi_\varepsilon^{-1}(1) \leq 2^{-\alpha}$. A standard calculus shows that if ω is given by $\omega(\varepsilon) := \varepsilon^{\theta/(p-\varepsilon)}$ for all $\varepsilon \in (0, p-1)$, then for any $\delta \in (0, p-1)$,

$$\sup_{\varepsilon \in [\delta, p-1)} \omega(\varepsilon) = (p-1)^\theta.$$

Thus, we conclude that the assumptions of Theorem 4.7 are satisfied with $q = p + \alpha$ and thus the required statement follows. □

Acknowledgments

The first and third authors were supported by the Shota Rustaveli National Science Foundation Grant (No. FR-18-2499) and the second author was supported by the National Science Centre, Poland, Grant 2015/17/B/ST1/00064.

The authors thank to the anonymous referee for remarks.

References

[1] T.C. Anderson, W. Damián, Calderón-Zygmund operators and commutators in spaces of homogeneous type: weighted inequalities, arXiv:1401.2061v1, 2014.

[2] T.C. Anderson, D. Cruz-Uribe, K. Moen, Logarithmic bump conditions for Calderón–Zygmund operators on spaces of homogeneous type, Publ. Mat. 59 (1) (2015) 17–43.

[3] N. Aronszajn, E. Gagliardo, Interpolation spaces and interpolation methods, Ann. Mat. Pura Appl. 68 (4) (1965) 51–117.

[4] S.V. Astashkin, K.V. Lykov, Jawerth-Milman extrapolation theory: some recent developments with applications, in: Functional Analysis, Harmonic Analysis, and Image Processing: a Collection of Papers in Honor of Björn Jawerth, in: Contemp. Math., vol. 693, Amer. Math. Soc., Providence, RI, 2017, pp. 7–53.

[5] S.V. Astashkin, M. Milman, Etrapolation: stories and problems, arXiv:2004.03655v1, 2020.

[6] A. Björn, J. Björn, Nonlinear Potential Theory on Metric Spaces, EMS Tracts Math., vol. 17, Eur. Math. Soc., Zürich, 2011.

[7] S.M. Buckley, Estimates for operator norms on weighted spaces and reverse Jensen inequalities, Trans. Am. Math. Soc. 340 (1) (1993) 253–272.

[8] A.P. Calderón, Intermediate spaces and interpolation, the complex method, Stud. Math. 26 (1964) 113–190.

[9] G. di Fratta, A. Fiorenza, V. Slastikov, An estimate of the blow-up of Lebesgue norms in the non-tempered case, preprint, 2020, 15 pp.

[10] A. Fiorenza, B. Gupta, P. Jain, The maximal theorem in weighted grand Lebesgue spaces, Stud. Math. 188 (2) (2008) 123–133.

[11] N. Fujii, Weighted bounded mean oscillation and singular integrals, Math. Jpn. 22 (5) (1977/1978) 529–534.

[12] L. Greco, T. Iwaniec, C. Sbordone, Inverting the p-harmonic operator, Manuscr. Math. 92 (1997) 249–258.

[13] T. Hytönen, A. Kairema, Systems of dyadic cubes in a doubling metric space, Colloq. Math. 126 (2012) 1–33.

[14] T. Hytönen, C. Pérez, E. Rela, Sharp reverse Hölder property for A^∞ weights on spaces of homogeneous type, J. Funct. Anal. 263 (12) (2012) 3883–3899.

[15] T. Iwaniec, C. Sbordone, On the integrability of the Jacobian under minimal hypotheses, Arch. Ration. Mech. Anal. 119 (1992) 129–143.

[16] B. Jawerth, M. Milman, Extrapolation theory with applications, Mem. Am. Math. Soc. 89 (440) (1991).

[17] A. Kamińska, M. Mastyło, Asymptotically isometric and isometric copies of ℓ_1 in some Banach function lattices, Comment. Math. 53 (2) (2013) 383–400.

[18] L.V. Kantorocich, G.P. Akilov, Functional Analysis, second edition, Pergamon Press, Oxford-Elmsford, N. Y., 1982.

[19] V. Kokilashvili, M. Krbec, Weighted Inequalities in Lorentz and Orlicz Spaces, World Scientific, Singapore, 1991.

[20] V. Kokilashvili, A. Meskhi, A note on the boundedness of the Hilbert transform in weighted grand Lebesgue spaces, Georgian Math. J. 16 (3) (2009) 547–551.

[21] V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, Integral Operators in Non-standard Function Spaces. Variable Exponent Hölder, Morrey-Campanato and Grand Spaces, Vol. 2, Birkhäuser/Springer, Heidelberg, 2016.

[22] S.G. Krein, Ju.I. Petunin, E.M. Semenov, Interpolation of Linear Operators, Nauka, Moscow, 1978, English transl. Transl. Math. Monogr., vol. 54, Amer. Math. Soc., Providence, RI, 1982.

[23] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces II, Springer-Verlag, Berlin, 1979.

[24] G.Ya. Lozanovskii, On some Banach lattices IV [English translation], Sib. Math. J. 14 (1973) 97–108.

[25] M. Mastyło, Interpolation of linear operators in Calderón-Lozanovskiii spaces, Comment. Math. Prace Mat. 26 (2) (1986) 247–256.

[26] M. Mastyło, E.A. Sánchez–Pérez, Köthe dual of Banach lattices generated by vector measures, Monatshefte Math. 173 (4) (2014) 541–557.

[27] V.I. Ovchinnikov, The method of orbits in interpolation theory, Math. Rep. 1 (2) (1984) 349–515.

[28] V.A. Shestakov, Transformations of Banach ideal spaces and interpolation of linear operators, Bull. Acad. Polon. Sci. Sér/Math. 29 (1981) (1982) 569–577 (Russian).

[29] A.C. Zaanen, Integration, North-Holland, Amsterdam, 1967.