Nonlinear Analysis 188 (2019) 70–79

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

The measure of noncompactness of multilinear operators

Vakhtang Kokilashvili [a,b,1], Mieczysław Mastyło [c,2], Alexander Meskhi [a,d,1]

[a] *Department of Mathematical Analysis, A. Razmadze Mathematical Institute, I. Javakhishvili Tbilisi State University, 6. Tamarashvili Str., 0177 Tbilisi, Georgia*
[b] *International Black Sea University, 3 Agmashenebeli Ave., Tbilisi 0131, Georgia*
[c] *Faculty of Mathematics and Computer Science, Adam Mickiewicz Univeristy, Poznań, Umultowska 87, 61-614 Poznań, Poland*
[d] *Department of Mathematics, Faculty of Informatics and Control Systems,Georgian Technical University, 77, Kostava St., Tbilisi, Georgia*

ARTICLE INFO

Article history:
Received 2 January 2019
Accepted 11 May 2019
Communicated by Vicentiu D Radulescu

MSC:
primary 46B70
42B30

Keywords:
Multilinear operators
Mutilinear Hilbert and Riesz transform
Measure of noncompactness
Compact operators
Approximation property

ABSTRACT

We investigate the multilinear variants of the quantities which measure the noncompactness of multilinear operators taking values in Banach spaces with the uniform approximation property. We show applications to multilinear variant of the Hilbert and Riesz transform on rearrangement invariant spaces. We derive lower estimates of the essential norm of these transforms. Along the way we obtain also as a by-product the lower estimates of the measure noncompactness of these transforms. As a consequence we conclude that the Hilbert and Riesz transforms are not compact multilinear transforms.

1. Introduction

In Banach space theory of operators the quantities which measure the noncompactness of operators play an important role. Throughout the paper we use standard notation from Banach space theory and operator theory. We recall that for a bounded subset A of a Banach space E the measure of noncompactness of A, denoted by $\gamma(A)$, is the infimum over all r such that A can be covered with a finite number of balls $B(x,r) := \{y \in E;\, \|x-y\|_E \le r\}$ with radius $r > 0$. We note that $\gamma(A)$ can be given by the formula:

$$\gamma(A) = \inf\{r > 0;\, A \subset K + rB_E,\ K \text{ compact}\},$$

where $B_E := B(0,1)$ denotes the closed unit ball of E.

E-mail addresses: vakhtang.kokilashvili@tsu.ge (V. Kokilashvili), mastylo@math.amu.edu.pl (M. Mastyło), alexander.meskhi@tsu.ge, a.meskhi@gtu.ge (A. Meskhi).

[1] The first and third authors were supported by the Shota Rustaveli National Science Foundation of Georgia (Project No. DI 18-118).

[2] The second author was supported by the National Science Centre of Poland, project 2015/17/B/ST1/00064.

https://doi.org/10.1016/j.na.2019.05.011

Let E and F be Banach spaces and let $\mathcal{L}(E,F)$ be the space of bounded linear operators between Banach spaces E and F. The measure of noncompactness of an operator $T \in \mathcal{L}(E,F)$ is given by $\gamma(T) = \gamma(T(B_E))$. Clearly, $\gamma(T) = 0$ if T is a compact linear operator and that

$$\gamma(S+T) \leq \gamma(S) + \gamma(T), \quad \gamma(\lambda T) = |\lambda|\gamma(T)$$

for every $S, T \in \mathcal{L}(E,F)$ and all scalars λ. These properties imply that $\gamma(T)$ induces the norm of the coset $[T]$ in the quotient Banach space $\mathcal{L}(E,F)/\mathcal{K}(E,F)$, where $\mathcal{K}(E,F)$ is the space of all compact linear operators from E into F.

The next important characteristic of the space $\mathcal{L}(E,F)$ is the so-called the essential norm defined for $T \in \mathcal{L}(E,F)$ by

$$\|T\|_e := \|[T]\|_{\mathcal{L}(E,F)/\mathcal{K}(E,F)} = \inf\bigl\{\|T-S\|_{E\to F};\ S \in \mathcal{K}(E,F)\bigr\}.$$

Observe that $\gamma(T) = \gamma(T-S)$ for arbitrary $S \in \mathcal{K}(E,F)$ and thus

$$\gamma(T) \leq \|T\|_e.$$

We note that Lebow and Schechter [8] proved that if the Banach space F has the λ-CAP, then for every Banach space E we have

$$\|T\|_e \leq \lambda\gamma(T), \quad T \in \mathcal{L}(E,F).$$

We recall that a Banach space E is said to have the λ-*c*ompact approximation property (λ-CAP for short) if for every compact subset $A \subset E$ and for every $\varepsilon > 0$ there exists a compact operator $S \in \mathcal{K}(E)$ such that

$$\sup_{x\in A} \|Sx - x\|_E \leq \varepsilon \quad \text{and } \|S - I\| \leq \lambda.$$

A Banach space E is said to have the bounded compact approximation property (BCAP) if it has the λ-CAP for some $\lambda \geq 1$. For an example of a Banach space without the BCAP we refer to [9, p. 111].

We point out that in the theory of Banach spaces an important role plays the bounded approximation property. Recall that the Banach space E is said to have the *bounded approximation property* (BAP for short) if there exists $\lambda > 0$ such that for every compact subset A and for every $\varepsilon > 0$ one can find a finite rank operator $V\colon E \to E$ with $\|V\| \leq \lambda$ and

$$\|Vx - x\|_E \leq \varepsilon, \quad x \in A.$$

Clearly that the essential norm of $T \in \mathcal{L}(E,F)$ is dominated by the measure of *non-approximability* $a(T)$ defined by

$$a(T) := \inf\bigl\{\|T-S\|_{E\to F};\ S \in \mathcal{L}(E,F),\ \mathrm{rank}(S) < \infty\bigr\}.$$

The aim of the paper is to define variants of measure of noncompactness in the setting of multilinear operators on the product of Banach spaces and show some applications to the multilinear variants of Hilbert and Riesz transforms. We emphasize that for the results similar to those obtained in this paper for linear operators in the classical and some non-standard Lebesgue spaces we refer to the monographs [3, Ch. V], [12] and references cited therein.

2. Results

We first fix some notation and collect a few definitions and results that will be needed in the sequel. Let $m \geq 2$ be an integer and let $X_1, \ldots, X_m$ and Y be Banach spaces. We equipped the product $X_1 \times \cdots \times X_m$

with the norm $\|(x_1,\dots,x_m)\| = \max_{1\le j\le m}\|x_j\|_{X_j}$. We denote by $\mathcal{L}_m(X_1\times\cdots\times X_m,Y)$ the Banach space of multilinear and continuous operators defined on $X_1\times\cdots\times X_m$ with values in Y equipped with the norm

$$\|T\|_{X_1\times\cdots\times X_m\to Y} := \sup\bigl\{\|T(x_1,\dots,x_m)\|_Y;\ (x_1,\dots,x_m)\in B_{X_1}\times\cdots\times B_{X_m}\bigr\}.$$

In the case Y is the scalar field ($\mathbb{R}$ or $\mathbb{C}$), we denote the space of multilinear forms by $\mathcal{L}_m(X_1\times\cdots\times X_m)$.

The measure of noncompactness of $T\in\mathcal{L}_m(X_1\times\cdots\times X_m)$ is given by

$$\gamma_m(T) := \gamma\bigl(T(B_{X_1}\times\cdots\times B_{X_m})\bigr).$$

An m-linear operator $T\colon X_1\times\cdots\times X_m\to Y$ is said to be compact if $T(B_{X_1}\times\cdots\times B_{X_m})$ is relatively compact in Y; the space of all such operators is denoted by $\mathcal{K}_m(X_1\times\cdots\times X_m,Y)$. Clearly that $\gamma_m(T)=0$ if T is compact.

For any $T\in\mathcal{L}_m(X_1,\dots,X_m,Y)$, we denote the image of T by $\mathrm{Im}(T)$. Since in general $\mathrm{Im}(T)$ is not a linear subspace of Y, we define $\mathrm{rank}(T)$ as the dimension of $[\mathrm{Im}(T)]$, where $[S]$ denotes the linear span of a subset S in a linear vector space W.

The space of all $T\in\mathcal{L}_m(X_1\times\cdots\times X_m,Y)$ such that $\mathrm{rank}(T)<\infty$ is denoted by $\mathcal{F}_m(X_1\times\cdots\times X_m,Y)$. We let

$$\|T\|_{e,m} := \inf\bigl\{\|T-S\|;\ S\in\mathcal{K}_m(X_1\times\cdots\times X_m,Y)\bigr\}$$

and

$$a_{(m)}(T) := \inf\bigl\{\|T-S\|;\ S\in\mathcal{F}_m(X_1\times\cdots\times X_m,Y)\bigr\}.$$

We have the following multilinear variant of the mentioned Lebow–Schechter result in the linear setting.

Lemma 2.1. *Let $X_1,\dots,X_m$ and Y be Banach spaces. Assume that Y has the λ-compact approximation property. Then for every $T\in\mathcal{L}_m(X_1\times\cdots\times X_m,Y)$,*

$$\|T\|_{e,m}\le\lambda\,\gamma_m(T).$$

Proof. Fix $\varepsilon>0$. There exists a finite set $K=\{y_1,\dots,y_k\}$ in Y such that

$$T(B_{X_1}\times\cdots\times B_{X_m})\subset K+(\gamma_m(T)+\varepsilon)B_Y.$$

By hypothesis there exists a compact operator $P\colon Y\to Y$ such that $\|P-I\|\le\lambda$ and

$$\|Py_j-y_j\|_Y<\varepsilon,\quad 1\le j\le k.$$

Let $x=(x_1,\dots,x_m)\in B_{X_1}\times\cdots\times B_{X_m}$ be given. Then there is a $y_j\in K$ such that

$$\|Tx-y_j\|_Y\le\gamma_m(T)+\varepsilon.$$

Combining the above estimates we get

$$\begin{aligned}\|(T-PT)x\|_Y &\le \|(I-P)(Tx-y_j)\|_Y+\|(I-P)y_j\|_Y\\ &\le \|I-P\|\,\|\|Tx-y_j\|_Y+\varepsilon\\ &\le \lambda(\gamma_m(T)+\varepsilon)+\varepsilon.\end{aligned}$$

Clearly that $PT\in\mathcal{K}_m(X_1\times\cdots\times X_m,Y)$ and so

$$\|T\|_{e,m}\le\lambda\gamma_m(T)+\varepsilon(\lambda+1).$$

Since ε is arbitrary, the required estimate follows. □

We will need the following technical result.

Proposition 2.2. *Let $X_1, \dots, X_m$ and Y be Banach spaces. Assume that Y has the bounded approximation property. Then for every $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$,*

$$\|T\|_{e,m} = a_{(m)}(T).$$

Proof. For given $\varepsilon > 0$, we can find $S \in \mathcal{K}_m(X_1 \times \cdots \times X_m, Y)$ such that

$$\|T - S\| < \|T\|_{e,m} + \varepsilon/2.$$

Our hypothesis $Y \in (BAP)$ easily yields that there exists $P \in \mathcal{F}_m(X_1 \times \cdots \times X_m, Y)$ such that $\|S - P\| \le \varepsilon/2$. Hence

$$\|T - P\| \le \|T - S\| + \|S - P\| \le \|T\|_{e,m} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get

$$a_{(m)}(T) \le \|T\|_{e,m}.$$

Clearly that $\|T\|_{e,m} \le a_{(m)}(T)$ and so the proof is complete. □

We will consider Banach function lattices. Recall that if $(\Omega, \mathcal{S}, \mu)$ is a complete and σ-finite measure space, then $L^0(\mu) := L^0(\Omega, \mathcal{S}, \mu)$ denotes the space of all equivalence classes of real-valued measurable functions on Ω. A *Banach function lattice* $(E, \|\cdot\|_E)$ on $(\Omega, \mathcal{S}, \mu)$ is defined to be a subspace of $L^0(\mu)$, endowed with a complete norm $\|\cdot\|_E$ such that there exists $u \in E$ with $u > 0$ a.e. It is also assumed that E is an ideal in $L^0(\mu)$, i.e., if $f \in L^0(\mu)$ satisfies $|f| \le |g|$ a.e., for some $g \in E$, then $f \in E$ and $\|f\|_E \le \|g\|_E$. A Banach function lattice E is said to be order continuous whenever for every non-negative sequence (f_n) in E satisfying $f_n \downarrow 0$ we have $\|f_n\|_E \to 0$. We will use the well known and easily verifiable fact that E is order continuous whenever for any sequence (f_n) satisfying $|f_n| \le g \in E$ for each $n \in \mathbb{N}$ and $f_n \to g$ μ-a.e., then $\|f_n - g\|_E \to 0$.

An important class of Banach function lattices is *rearrangement invariant* (r.i. for short) spaces. We recall its definition below. Given $f \in L^0(\mu)$, its distribution function is defined by $\mu_f(\lambda) = \mu\{t \in \Omega; |f(t)| > \lambda\}$, $\lambda \ge 0$. Then a Banach lattice E on $(\Omega, \mathcal{S}, \mu)$ is a r.i. space if $g \in E$ and $\|f\|_E = \|g\|_E$ whenever $\mu_f = \mu_g$ and $f \in E$. It is well known [1,7] that for any r.i. space E we have

$$L_1 \cap L_\infty \subset E \subset L_1 + L_\infty$$

and so in particular this implies that any function $f \in E$ is locally integrable. Clearly, for a r.i. space E, $\|\chi_A\|_E$ depends only on $\mu(A)$ for any measurable set A of finite measure. If $(\Omega, \mathcal{S}, \mu)$ is an atomless measure space, then for every $t \in \mathbb{R}_+$ with $t \le \mu(\Omega)$, we may define the function ϕ_E by $\phi_E(t) = \|\chi_A\|_E$, where A is any measurable set with $\mu(A) = t$. This function is called the *fundamental function* of E.

Let X be a Hausdorff topological space and $\mathcal{B}(X)$ the Borel subsets of X. Let Σ be a σ-algebra containing $\mathcal{B}(X)$ and $\mu \colon \Sigma \to [0, \infty]$ be finitely additive. We recall that $A \in \Sigma$ is inner regular if

$$\mu(A) = \sup\{\mu(K);\ K \subset A,\ K \text{ compact}\};$$

$E \in \Sigma$ is outer regular if $\mu(A) = \inf\{\mu(G);\ G \supset A,\ G \text{ open}\}$. A is regular if A is both inner and outer regular. A Borel measure is a measure defined on the Borel sets, which is finite on compact sets. A Borel measure is regular if every set is outer regular and every open set is inner regular.

We also will work on metric measure spaces. We refer to [2,6] for the theory and application of these spaces in potential theory. We recall that if (X, d) is a metric space and μ is a measure on a σ algebra $\mathcal{B}(X)$

of Borel sets, then the measure space $(X, \mathcal{B}(X), \mu)$ denoted by (X, d, μ) is called a metric measure space. As usual a ball with a center at $x \in X$ and radius $r > 0$ in a metric space (X, d) is denoted by

$$B := B(x, r) = \{y \in X; d(x, y) < r\}.$$

To avoid trivial measures we will always assume throughout the paper that $0 < \mu(B) < \infty$ for every ball B. Consequently, μ is σ-finite measure.

Lemma 2.3. *Let $X_1, \dots, X_m$ be Banach spaces and let E be order continuous Banach function lattices on a metric measure space (X, d, μ). If $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, E)$ with $N = \operatorname{rank}(T) < \infty$, then for every $a \in X$ and $\varepsilon > 0$, there exist $S \in \mathcal{L}_m(X_1 \times \cdots \times X_m, E)$ and $0 < r < R < \infty$ such that $\operatorname{rank}(S) \le N$,*

$$\|T - S\|_{X_1 \times \cdots \times X_m \to E} \le \varepsilon$$

and $\operatorname{supp}(S(x)) \subset B(a, R) \setminus B(a, r)$ for all $x \in X_1 \times \cdots \times X_m$.

Proof. Let $V = [\operatorname{Im}(T)] \subset E$ with $\dim(V) = N$. We use Auerbach's Lemma which states, that there are unit basis vectors $f_1, \dots, f_N \in V$ and unit vectors $f_1^*, \dots, f_N^* \in V^*$ such that

$$\langle f_i, f_j^* \rangle = \delta_{ij}, \quad 1 \le i, j \le N.$$

By the Hahn–Banach Theorem, it follows that there exists $\varphi_j^* \in E^*$ such that $\varphi_j^*|_V = f_j^*$. Since $Tx \in V$ for all $x \in X_1 \times \cdots \times X_m$,

$$Tx = \sum_{j=1}^{N} \varphi_j^*(Tx) f_j = \sum_{j=1}^{N} \varphi_j^* \circ T(x) f_j.$$

Hence

$$T = \sum_{j=1}^{N} B_j \otimes f_j,$$

where $B_j := \varphi_j \circ T \in \mathcal{L}_m(X_1 \times \cdots \times X_m)$ with $\|B_j\| \le \|T\|$ for each $1 \le j \le m$. This implies that, for every $x = (x_1, \dots, x_m) \in X_1 \times \cdots \times X_m$, we have

$$\sum_{j=1}^{N} |B_j(x)| \le N\|T\| \, \|x_1\|_{X_1} \cdots \|x_m\|_{X_m}.$$

Now we fix two positive sequences $\{r_n\}$ and $\{R_n\}$ such that $r_n < R_n$ for each $n \in \mathbb{N}$ and $r_n \downarrow 0$ and $R_n \uparrow \infty$. For a given $a \in X$ and for each $n \in \mathbb{N}$, we let $A_n := B(a, R_n) \setminus B(a, r_n)$. Combining $f\chi_{A_n} \to f$ μ-a.e. with order continuity of E, it follows that for all $f \in E$, we have

$$\|f - f\chi_{A_n}\|_E \to 0 \quad \text{as } n \to \infty.$$

This implies that there exist positive r and R with $r < R$ such that

$$\|f_j - f_j\chi_{B(a,R)\setminus B(a,r)}\|_E \le \frac{\varepsilon}{N\|T\|}, \quad 1 \le j \le N.$$

We let $g_j := f_j\chi_{B(a,R)\setminus B(a,r)}$ for each $1 \le j \le N$ and define

$$Sx := \sum_{j=1}^{N} B_j(x) g_j, \quad x \in X_1 \times \cdots \times X_m.$$

Clearly that $S \in \mathcal{L}_m(X_1 \times \cdots \times X_m, E)$, $\operatorname{rank}(S) \leq N$ and $\operatorname{supp}(S(x)) \subset B(a,R) \setminus B(a,r)$ for all $x \in X_1 \times \cdots \times X_m$. Combining the above estimates yields

$$\|Tx - Sx\|_E \leq \sum_{j=1}^{N} |B_j(x)| \|f_j - g_j\|_E \leq \frac{\varepsilon}{N\|T\|} \sum_{j=1}^{N} |B_j(x)| \leq \varepsilon \|x_1\|_{X_1} \cdots \|x_m\|_{X_m}$$

for all $x = (x_1, \ldots, x_m) \in B_{X_1} \times \cdots \times B_{X_m}$. Thus $\|T - S\| \leq \varepsilon$ as required. $\square$

We show that a large class of Banach lattices on some metric measure spaces have compact approximation property. Before we state the result we need to introduce some more notion and definitions. Let (X,d,μ) be a metric measure space. A μ-measurable function $f\colon X \to \mathbb{R}$ is said to be locally integrable ($f \in L^1_{\mathrm{loc}}(\mu)$ for short) if f is integrable over any balls of X. If (X,d,μ) is a measure metric space such that $0 < \mu(B) < \infty$, we define the Hardy–Littlewood operator M_μ by

$$M_\mu f(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f|\, d\mu, \qquad f \in L^1_{\mathrm{loc}}(\mu),$$

where the supremum is taken over all balls B containing x.

It is known (see e.g., [1], P. 154) that for example, if $X = \mathbb{R}^n$ and μ is a Lebesgue measure, then the maximal operator M_μ is bounded in r.i. space E if and only if upper index of E is less than 1.

A metric measure space (X,d,μ) is said to be a *Vitali space* if all balls in X have finite measure and if for every set A in X and every family $\mathcal{B}$ of closed balls in X satisfying

$$\inf\bigl\{r > 0;\ B(x,r) \in \mathcal{F}\bigr\} = 0 \quad \text{for } x \in A,$$

there is a countable disjoint subfamily $\{B_n\}$ of $\mathcal{B}$ such that

$$\mu\Bigl(A \setminus \bigcup_n B_n\Bigr) = 0.$$

It is well known that if (X,d,μ) with μ a doubling measure, then it follows by the well known Vitali's theorem that X is a Vitali space (see [6, Theorem 1.6]). We note that if μ is a Radon measure on $\mathbb{R}^n$, then $(\mathbb{R}^n, \mu)$ is also a Vitali space (see [11, Theorem 2.8]).

We recall that a measure μ is doubling if there is a constant $C_\mu > 0$ such that for all balls $\mu(2B) \leq C_\mu \mu(B)$, where $2B(x,r) = B(x,2r)$.

Let X be locally compact Hausdorff space. We denote by $C_c(X)$ the space of all real-valued continuous functions on X which have compact support. In what follows we will need the following proposition.

Proposition 2.4. *Let X be a locally compact Hausdorff space and let μ be a Radon measure on $\mathcal{B}(X)$ and let $E \subset L^0(\mu)$ be order continuous Banach function lattice such that $\chi_A \in E$ for every μ-finite Borel set. Then $C_c(X)$ is dense in E.*

Proof. Clearly, the order continuity of E implies that the set of simple functions is dense in E. If $\{\varepsilon_n\}$ is a positive sequence with $\varepsilon_n \downarrow 0$ and s is a simple function in E, then it follows from Luzin's theorem that there exist a sequence $\{f_n\}$ of functions in $C_c(X)$ such that except on a measurable set A_n with $\mu(A_n) < \varepsilon_n$, $|f_n| \leq \|s\|_\infty$. This implies that, for each n, we have

$$\|f_n - s\|_E \leq \|s\|_\infty \|\chi_{A_n}\|_E$$

and so $f_n \to s$ in E by order continuity. $\square$

We are ready to prove the following result which shows that a large class of Banach lattices on metric measure spaces have compact approximation property.

Theorem 2.5. *Let (X, d, μ) be a locally compact Vitali measure metric space such that μ is a regular measure with $\mu(\partial B) = 0$ for every ball in X. Assume that E is an order continuous Banach function lattice in $L^0(X, \mu)$ with $\chi_A \in E$ for any μ-measurable set A and that the Hardy–Littlewood maximal operator M_μ is bounded in E. Then E has λ-compact approximation property with $\lambda = \|M_\mu\| + 1$.*

Proof. Fix $\varepsilon > 0$ and let $f_1, \dots, f_N$ be given functions in E. By Proposition 2.4, it follows that $C_c(X)$ is dense in E, we may assume that f_j are continuous functions with compact supports for each $1 \le j \le N$. Let K be a compact set outside which all f_j vanish. The uniform continuity on K of each f_j yields that there exists $\delta > 0$ such that, for every $x, y \in K$ with $d(x, y) < \delta$, we have

$$|f_j(x) - f_j(y)| < \frac{\varepsilon}{2\|\chi_E\|_E}, \quad 1 \le j \le N.$$

Since (X, d, μ) is a Vitali space, it follows that the family $\mathcal{B}$ of closed balls,

$$\mathcal{B} := \big\{\bar{B}(x, r);\, x \in K,\, B(x, r) \subset \operatorname{int} K,\, r \in (0, \delta/2)\big\} \cup \big\{\bar{B}(x, r);\, x \in \partial K,\, r \in (0, \delta/2)\big\}$$

contains a countable disjoint subfamily $\{\bar{B}_n\}_{n=1}^\infty$ such that

$$\mu\Big(K \setminus \bigcup_{n=1}^\infty \bar{B}_n\Big) = 0.$$

We note that $d(x, y) < \delta$ for all $x, y \in B_n$ and each $n \in \mathbb{N}$. Since every ball with center at $z \in \partial K$ contains a point from K^c, it follows by continuity of f_j that $f_j(z) = 0$ for all $z \in \partial K$ and for each $1 \le j \le N$. Combining we obtain that, for each $n \in \mathbb{N}$ and each $1 \le j \le N$,

$$|f_j(x) - f_j(y)| \le \frac{\varepsilon}{2\|\chi_K\|_E}\, \chi_K(x), \quad x \in B_n \cap K,\ y \in B_n \setminus K.$$

Now we define a linear mapping S on E by

$$Sf := \sum_{n=1}^\infty \Big(\frac{1}{\mu(B_n)} \int_{B_n} f\, d\mu\Big) \chi_{K \cap B_n}, \quad f \in E.$$

Observe that for every $x \in X$ we have

$$|Sf(x)| \le \sum_{n=1}^\infty \Big(\frac{1}{\mu(B_n)} \int_{B_n} |f|\, d\mu\Big) \chi_{B_n}(x) \le M_\mu f(x)$$

and so $S\colon E \to E$ is bounded with $\|S\|_{E\to E} \le \|M_\mu\|_{E\to E}$. Consequently, $\|S - I\|_{E\to E} \le \|M_\mu\|_{E\to E} + 1$.

We claim that S is a compact operator. To see this observe that the sequence $\{S_n\}$ of finite dimensional operators given by

$$S_n f := \sum_{j=1}^n \Big(\frac{1}{\mu(B_j)} \int_{B_j} f\, d\mu\Big) \chi_{K \cap B_j}, \quad f \in E$$

satisfies $S_n f \to Sf$ and $|S_n f| \le S|f| \in E$. This implies by order continuity of E that

$$\|S_n f - Sf\|_E \to 0 \quad \text{as } n \to \infty.$$

To finish the proof it is enough to show that

$$\|Sf_j - f_j\|_E \le \varepsilon, \quad 1 \le j \le N.$$

To see this we recall that $d(x,y) < \delta$ for all $x, y \in B_n$ and each $n \in \mathbb{N}$. Since $(K \cap B_i) \cap (K \cap B_k) = \emptyset$ for each $i \neq k$ and $\mu(K \setminus \bigcup_{n=1}^{\infty} B_n) = 0$, $f_j = \sum_{n=1}^{\infty} f_j \chi_{K \cap K_n}$ a.e. for each $1 \le j \le N$. Combining these with the second estimate at the beginning of the proof we obtain that, for almost all $x \in X$ and for each $1 \le j \le N$,

$$\begin{aligned}
|Sf_j(x) - f_j(x)| &= \left| \sum_{n=1}^{\infty} \left(\frac{1}{\mu(B_n)} \int_{B_n} f_j \, d\mu \right) \chi_{K \cap B_n}(x) - f_j(x) \right| \\
&\le \sum_{n=1}^{\infty} \left(\frac{1}{\mu(B_n)} \int_{B_n} |f_j(y) - f_j(x)| \, d\mu(y) \right) \chi_{K \cap B_n}(x) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\mu(B_n)} \int_{K \cap B_n} |f_j(x) - f_j(y)| \, d\mu(y) \right) \chi_{K \cap B_n}(x) \\
&\quad + \sum_{n=1}^{\infty} \left(\frac{1}{\mu(B_n)} \int_{B_n \setminus K} |f_j(x) - f_j(y)| \, d\mu(y) \right) \chi_{K \cap B_n}(x) \\
&\le \frac{\varepsilon}{\|\chi_K\|_E} \chi_K(x)
\end{aligned}$$

and so the required estimate follows. □

We finish with the following an obvious remark that the condition $\mu(\partial B) = 0$ for any ball B holds in a wide class of metric measure spaces (X, d, μ). For example, this is true whenever the function $(0, d_X) \ni r \mapsto \mu(B(x,r))$ is continuous for every x, then $\mu(\partial B(x,r)) = 0$, where $d_X := \sup_{x,y \in X} d(x,y)$. In fact we have

$$\partial B(x,r) \subset B(x, r + \varepsilon_n) \setminus B(x, r - \varepsilon_n)$$

for any sequence $\{\varepsilon_n\}$ in $(0, r - d_X)$ with $\varepsilon_n \to 0$ as $n \to \infty$ and whence,

$$\mu(\partial B(x,r)) \le \mu(B(x, r + \varepsilon_n)) - \mu(B(x, r - \varepsilon_n)), \quad n \in \mathbb{N}.$$

Clearly this yields $\mu(\partial B(x,r)) = 0$.

3. Application to multilinear variant of Hilbert and Riesz transforms

In this section we apply the derived results to obtain the lower estimate of the essential norm for multilinear variants of Hilbert and Riesz transforms bounded in r.i. Banach spaces.

For each positive integer m we define a variant of multilinear Hilbert transform H_m on $L^1_{\mathrm{loc}}(\mathbb{R}^m, \lambda_m)$ given by the formula:

$$H_m(f_1, \ldots, f_m)(x) := \text{p.v.} \int_{\mathbb{R}^m} \frac{f_1(t_1) \cdots f_m(t_m)}{\big((x - t_1) + \cdots + (x - t_m)\big)^m} \, d\lambda_m, \quad x \in \mathbb{R},$$

where λ_m is the Lebesgue measure in $\mathbb{R}^m$.

Theorem 3.1. *Let $X_1, \ldots, X_m$ and E be r.i. Banach spaces on $\mathbb{R}$ such that $H_m \in \mathcal{L}_m(X_1 \times \cdots \times X_m, E)$. If E has bounded approximation property and $\varphi_{X_1}(t) \cdots \varphi_{X_m}(t) \le C \varphi_E(t)$ for all $t > 0$ and some $C > 0$, then*

$$\|H_m\|_{e,m} \ge \frac{1}{C(2m)^m}.$$

Proof. Let $\delta > \|H_m\|_{e,m}$. From Proposition 2.2, it follows that $\delta > a_{(m)}(T)$. Thus, by the definition, there exists $T \in \mathcal{F}_m(X_1 \times \cdots \times X_m, E)$ such that

$$\|H_m - T\| < \delta.$$

Applying Lemma 2.3 with $a = 0$, we conclude that there are positive number $\beta > 0$ and $S \in \mathcal{F}_m(X_1 \times \cdots \times X_m, E)$ such that

$$\|T - S\| < \frac{1}{2}\big(\delta - \|H_m - S\|\big)$$

and for all $f \in X_1 \times \cdots \times X_m$,

$$\operatorname{supp}(Sf) \subset \mathbb{R} \setminus (-\beta, \beta).$$

Hence

$$\|(H_m - S)f\|_E \le \delta \|f_1\|_{X_1} \cdots \|f_m\|_{X_m} \tag{$*$}$$

and so for every $\tau \in (0, \beta)$,

$$\|\chi_{(-\tau,\tau)} H_m f\|_E \le \delta \|f_1\|_{X_1} \cdots \|f_m\|_{X_m}.$$

Now observe that for each $1 \le j \le m$ and for all $t_j \in (0, \tau)$, $x \in (-\tau, 0)$, we have that $0 < (t_j - x) < 2\tau$. Then from $(*)$, taking $f_j = \chi_{(0,\tau)}$ for all $j \in \{1, \ldots, m\}$, it follows that

$$\begin{aligned}
\delta\, \|\chi_{(0,\tau)}\|_{X_1} \cdots \|\chi_{(0,\tau)}\|_{X_m} &\ge \|\chi_{(-\tau,0)} H_m \chi_{(0,\tau)}\|_E \\
&\ge \left\| \chi_{(-\tau,0)} \int_{(0,\tau)^m} \big((x - t_1) + \cdots + (x - t_m)\big)^{-m}\, d\lambda_m \right\|_E \\
&\ge \frac{1}{(2m\tau)^m} \left(\int_{(0,\tau)^m} d\lambda_m \right) \|\chi_{(-\tau,0)}\|_E.
\end{aligned}$$

By hypothesis on fundamental functions we get

$$\delta \ge \frac{1}{C(2m)^m}.$$

Since $\delta > \|H_m\|_{e,m}$ was arbitrary, the required estimate follows. □

Further, we are also interested in the estimate of the essential norm for the multilinear Riesz transform. Let us recall (see, e.g., [5]) that for each positive integer m and for each $1 \le i \le m$, an m-linear i-th Riesz transform $R_i^{(m)}$ on $L^1_{\mathrm{loc}}(\mathbb{R}^{mn}, \lambda_{mn})$ is defined by the formula:

$$R_i^{(m)}(f_1, \ldots, f_m)(x) := \text{p.v.} \int_{(\mathbb{R}^n)^m} \frac{\sum_{j=1}^m (x_i - (y_j)_i)}{\left(\sum_{j=1}^m |x - y_j|^2\right)^{\frac{mn+1}{2}}} f_1(y_1) \cdots f_m(y_m)\, d\lambda_{mn}, \quad x \in \mathbb{R}^n,$$

where λ_{mn} is the Lebesgue measure in $\mathbb{R}^{mn}$. Here x_i and $(y_j)_i$ are i-th coordinates of x and y_j, respectively.

We note that if $n = m = 1$, then $R_i^{(1)}$ is the linear Hilbert transform H. Now we are ready to prove the following statement:

Theorem 3.2. *Let $X_1, \ldots, X_m$ and E be rearrangement invariant Banach spaces defined on $\mathbb{R}^n$ such that $R_i^{(m)} \in \mathcal{L}_m(X_1 \times \cdots \times X_m, E)$ for each $i \in \{1, \ldots, n\}$. If E has bounded approximation property and $\varphi_{X_1}(t) \cdots \varphi_{X_m}(t) \le C\varphi_E(t)$ for all $t > 0$ and some $C > 0$, then the following estimate holds for the sum of the m-linear Riesz transforms $R^{(m)} := \sum_{i=1}^n R_i^{(m)}$,*

$$\|R^{(m)}\|_{e,m} \ge \frac{1}{C(2m)^{mn} n^{(mn+1)/2}}.$$

Proof. Following to the proof of Theorem 3.1 we take $\delta > \|H_m\|_{e,m}$. Arguing as in the proof of Theorem 3.1 we find that the inequality

$$\|(R^{(m)} - S)f\|_E \le \delta \|f_1\|_{X_1} \cdots \|f_m\|_{X_m}$$

holds for all $f = (f_1, \dots, f_m) \in X_1 \times \cdots \times X_m$, with $\operatorname{supp}(Sf) \subset \mathbb{R}^n \setminus Q(0,\beta)$, where $Q(0,\beta)$ is the cube with center 0 and side length 2β.

Let $0 < \tau < \beta$ and let $Q := Q(0,\tau)$. Following [4, p. 417], denote by Q' the cubes touching Q and having the same side length as Q, and such that $x_i \geq (y_j)_i$ for any $x \in Q$ and $y_j \in Q'$. Then, for every $x \in Q$ and $f_k = \chi_{Q'}$ for each $1 \leq k \leq m$, we have

$$\begin{aligned} R^{(m)}(f_1, \dots, f_m)(x) &\geq \sum_{i=1}^{n} \int_{(Q')^m} \frac{\sum_{j=1}^{m} (x_i - (y_j)_i)}{\left(\sum_{j=1}^{m} |x - y_j|\right)^{mn+1}} f_1(y_1) \cdots f_m(y_m)\, d\lambda_{mn} \\ &= \int_{(Q')^m} \frac{\sum_{j=1}^{m} \sum_{i=1}^{n} (x_i - (y_j)_i)}{\left(\sum_{j=1}^{m} |x - y_j|\right)^{mn+1}}\, d\lambda_{mn} \\ &\geq n^{-1/2} \int_{(Q')^m} \frac{\sum_{j=1}^{m} |x - y_j|}{\left(\sum_{j=1}^{m} |x - y_j|\right)^{mn+1}}\, d\lambda_{mn} \\ &\geq n^{-1/2} \int_{(Q')^m} \frac{1}{\left(\sum_{j=1}^{m} |x - y_j|\right)^{mn}}\, d\lambda_{mn} \\ &\geq \frac{(2\tau)^{mn} n^{-1/2}}{(4m\sqrt{n}\tau)^{mn}} = \frac{1}{(2m)^{mn} n^{(mn+1)/2}}. \end{aligned}$$

Hence, taking $f_j = \chi_{Q'}$, $j \in \{1, \dots, m\}$, we find that

$$\delta \, \|\chi_{Q'}\|_{X_1} \cdots \|\chi_{Q'}\|_{X_m} \geq \left\| \chi_Q R^{(m)} \chi_{Q'} \right\|_E \geq \frac{1}{(2m\sqrt{n})^{mn}} \|\chi_Q\|_E.$$

Thus we have

$$\delta \geq \frac{1}{C(2m)^{mn} n^{(mn+1)/2}}$$

and the proof is complete. □

We conclude with the following remark that order continuous r.i. spaces have bounded approximation property (see [10]).

References

[1] C. Bennett, R. Sharpley, Interpolation of operators, in: Pure and Applied Mathematics, vol. 119, Academic Press, Boston, 1988.

[2] A. Björn, J. Björn, Nonlinear potential theory on metric spaces, in: EMS Tracts in Mathematics, vol. 17, European Math. Soc., Zurich, 2011.

[3] D.E. Edmunds, W.D. Evans, Spectral Theory and Differential Operators, Oxford Univ. Press, Oxford, 1987.

[4] J. García-Cuerva, J.L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North Holland, Amsterdam, 1985.

[5] L. Grafakos, R.H. Torres, Multilinear Calderón-Zygmund theory, Adv. Math. 165 (1) (2002) 124–164.

[6] J. Heinonen, Lectures on Analysis on Metric Spaces, Universitext, Springer-Verlag, New York, 2001.

[7] S.G. Krein, Ju I. Petunin, E.M. Semenov, Interpolation of linear operators, in: AMS Translations of Math. Monog., vol. 54, Providence, 1982.

[8] A. Lebow, M. Schechter, Semigroups of operators and measures of noncompactness, J. Funct. Anal. 7 (1971) 1–26.

[9] J. Lindenstrauss, L. Tzafriri, Classical banach spaces II, in: Function Spaces, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[10] M. Mastyło, Interpolation of compact operators, Funct. Approx. Comment. Math. 26 (1998) 293–311.

[11] P. Mattila, Geometry of sets and measures in Euclidean spaces, in: Fractals and Rectifiability, in: Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995.

[12] A. Meskhi, Measure of Non-Compactness for Integral Operators in Weighted Lebesgue Spaces, Nova Science Publishers, New York, 2009.